Firm and Worker Dynamics in a Frictional Labor Market*

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Abstract

This paper develops a continuous-time random-matching model of a frictional labor market with firm and worker dynamics. Multi-worker firms choose whether to shrink or expand their employment in response to productivity shocks to their decreasing returns to scale technology. Growing entails posting costly vacancies, which are filled either by the unemployed or by employees poached from other firms. Firms also choose optimally when to enter and exit the market. Tractability is obtained by showing that, under a parsimonious set of assumptions, all worker and firm decisions can be characterized by comparisons between marginal surpluses which only depend on productivity and size. A version of the model calibrated to the U.S. economy reproduces key patterns of job reallocation and worker flows, in particular net poaching across the firm size, age, and productivity distribution.

Keywords: Decreasing Returns to Scale, Firm Dynamics, Net Poaching, On the Job Search, Unemployment, Vacancies, Worker Flows.

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1 Introduction

Aggregate production in the economy is divided into millions of firms, each facing large idiosyncratic fluctuations in its productivity and demand. As a result of this vast, stochastic, micro heterogeneity, there exist a large amount of turnover of firms and reallocation of workers across firms. The lack of perfect information, on both sides of the labor market, about suitable and profitable firm-worker matches means that search is a key feature of the labor market and that labor reallocation is frictional. While unemployment is a necessary consequence of frictional reallocation, notably a significant share of worker flows is the result of direct job-to-job transitions.

Understanding the process of the frictional reallocation of labor across heterogeneous production units is crucial for at least in three different reasons. First, in the long run, an effective reallocation of labor away from shrinking or unproductive firms and toward growing and productive firms enhances aggregate productivity and growth. Second, in the short run, the propagation of sectoral or aggregate shocks depends on how quickly labor flows across sectors/firms and between unemployment and employment. Third, from a normative perspective, policies that offer subsidies to jobless workers, protect employment, or advantage particular firms/sectors are necessarily forced to trade-off these objectives against the welfare losses from frictional reallocation.

A framework that offers a comprehensive way to address these issues both conceptually and quantitatively should therefore have three features: (i) a frictional labor market; (ii) a theory of the firm (i.e., its boundaries) and of firm dynamics (entry, growth, endogenous separations, exit); (iii) a theory of worker flows not limited to transitions from/into unemployment but inclusive of on-the-job search and poaching. In this paper, we present the first model with these characteristics based on random search.[1]

In our model, the frictions originate from random search, an aggregate matching function and costly vacancy posting. A firm is an owner of a production technology with decreasing returns to scale (in this sense, we do not distinguish between firm and establishment). Firms choose optimally whether to enter and exit the market. An incumbent firm that receives a positive shock to productivity expands by posting costly vacancies, which are—with some probability—filled either by unemployed workers, or by workers poached from other firms. The 'adjustment cost' to a firm is therefore endogenous: relative to its competitors, a firm that is more attractive to workers at other firms will require less vacancies in order to hire a worker. Workers choose whether to accept a job offer by comparing the value of their current option (unemployment or employment in their firm) to the value of working at the new firm.

[1] Directed search models with these features already exist, although they have shortcomings as we discuss in the literature section.

What determines these values to workers will be, in general, a complicated function of the workforce at the firm, the firm's productivity and the option value of future search.

The first contribution of our paper is to set out a sufficient and parsimonious set of assumptions regarding inter- and intra-firm bargaining and surplus sharing such that the firm-level state-vector can be reduced to only the number of workers in a firm and the firm's productivity level. Three key assumptions allow this state-space reduction: (i) lack of commitment in hiring, firing, and quit decisions; (ii) wage contract renegotiation only by mutual consent; (iii) unrestricted transfers of value between the firm and each of its worker during wage negotiations. With a parsimonious state vector we are then unrestricted in terms of permanent, persistent and transitory features of firm heterogeneity, freeing us up to construct a quantitative model of firm dynamics.[2]

Two other ingredients help achieve a tractable set up. First, we work in continuous time such that in a small interval of time only one random event may occur. For example, a firm only need deal with one of its workers meeting another firm, or a posted vacancy meeting an employed worker, but not both. Second, we take the continuous limit of a discrete workforce. As a result, this negotiation with only one worker at the margin of the firm, becomes truly marginal in the sense of a continuous partial derivative of firm values with respect to employment.[3].

The outcome is a model in which firms hire by posting vacancies until the expected marginal surplus from hiring equals the marginal cost of recruiting inputs. Firms (frictionlessly) fire workers until the marginal surplus of a worker equals the value of unemployment, and exit when the total value of production is less than the total value of unemployment to all its employees. Meanwhile, on-the-job search gives workers the opportunity to switch employers when they meet a vacancy posted by a higher marginal surplus firm.

The model generates the standard moments common to both models of firm dynamics (e.g. size distribution, job creation / destruction rates, entry and exit rates) and single worker models of labor-search (e.g. unemployment rate, UE flows). We calibrate out model to these moments for the U.S. economy.

The added value of our framework is that it generates a high-frequency panel of matched employer-employee data capturing worker flows in the cross-section of firms. We can therefore speak directly to two new set of facts based on such data. First, worker flows and net poaching by

[2]For example, we could arbitrarily include firm-level heterogeneity in the extent of decreasing returns, costs of vacancy posting, or sectoral heterogeneity in the size of idiosyncratic shocks.

[3]This echoes the application of Stole and Zwiebel (1996) bargaining in Acemoglu and Hawkins (2014) and Elsby and Michaels (2013). The objects that enter the surplus splitting rule determined by Nash bargaining are partial derivatives of firm value with respect to employment.

	All firms	A. Size		B. Size and Age		C. Productivity	
		Small n<50	**Large** n≥500	**Small, Young** Age≤10	**Large, Old** Age>10	**Low** Q1	**High** Q5
Total worker + job reallocation rate	31.7	37.4	27.7	44.6	26.5	30.8	24.2
Percent due to only worker reallocation (poaching)	52.4	47.0	56.0	47.9	56.0	49.4	50.4
Percent due to job creation / destruction	47.6	53.0	44.0	52.1	44.0	50.6	49.6

Table 1: Decomposing reallocation

Notes: (i) All rates are quarterly. (ii) Data are computed using Table 2, see notes therein. *Total reallocation rate* is the sum of all flow rates: $EE_{hir} + NE + EE_{sep} + EN$. *Percent due to only worker reallocation* gives the fraction of the total reallocation rate due to $EE_{hir} + EE_{sep}$. *Percent due to job creation / destruction* gives the fraction of the total reallocation rate due to $NE + EN$.

firm characteristics such as size, age (Haltiwanger, Hyatt, Kahn, and McEntarfer, 2018) and productivity (Haltiwanger, Hyatt, and McEntarfer, 2018) (see Table 2 and Table 1). Second, the response of worker flows and net poaching to shocks to firm value added documented in preliminary work by Bagger, Fontaine, Galenianos, and Trapeznikova (2018) (see Table 3).

Three take aways from these data. Poaching accounts for about half of total worker reallocation (Table 1). Poaching rates vary systematically across the firm distribution (larger for small firms—which the authors take as evidence a size ladder—and larger for high productivity firms—which the authors take as evidence of a productivity ladder) (Table 2). Increasing poaching hires and decreasing poached quits is the main avenue through which firms grow following positive shocks (Table 3). With both off- and on-the-job search and multi-worker firms our model is able to generate the moments studied in these papers.

Finally, we perform two quantitative counterfactual exercises.[4] First, we consider an economy in which multi-worker firms operate in a competitive labor market but face a convex adjustment cost (a la Clementi and Palazzo (2010)). We calibrate this alternative model to the same data on firm dynamics and compare the implications for output. Second, we consider an economy in which single-worker forms operate in a frictional labor market (a la Postel-Vinay and Robin (2002)). We calibrate this alternative model to the same data on worker flows and, again, compare the implications for output. These two exercises aim to understand the two contributions of our framework: (i) random matching with on-the-job search as micro-founded frictions in a multi-worker firm dynamics model, (ii) multi-worker firms in a random matching model with on-the-job search.

[4]In progress

		All firms	A. Size		B. Size and Age		C. Productivity	
			Small $n<50$	**Large** $n\geq500$	**Small, Young** Age$\leq$10	**Large, Old** Age>10	**Low** Q1	**High** Q5
A. Gross flows								
Hires								
Poaching	EE_{hir}	8.3	8.9	7.7	11. 1	7.4	7.2	6.5
Non-employment	NE	7.7	10.2	6.2	12.2	5.9	7.7	6.0
Separations								
Poaching	EE_{sep}	8.3	8.7	7.8	10.3	7.5	8.0	5.7
Non-employment	EN	7.4	9.6	6.0	11.0	5.7	7.9	6.0
B. Net flows								
Poaching	$EE_{hir} - EE_{sep}$	0.0	0.2	-0.1	0.8	-0.1	-0.8	0.8
Non-employment	$NE - EN$	0.3	0.5	0.2	1.2	0.2	-0.2	0.0
C. Total net flows								
Poaching + Non-employment		0.29	0.78	0.06	1.98	0.07	-1.00	0.80

Table 2: Average rates of worker and job flows by firm type

Notes: (i) All rates are quarterly. (ii) Data for the first five columns are the authors' calculations from the data made publicly available by Haltiwanger, Hyatt, Kahn, and McEntarfer (2018), who compute job and worker flows using LEHD data. The data consist of time series of quarterly rates computed as the relevant flow divided by total employment in the cell. Entries in the table consist of unweighted time-series averages of these quarterly rates, computed over 2002-2007. (iii) Data for the last two columns come from Table 3 of Haltiwanger, Hyatt, and McEntarfer (2018), which—similar to our computations—gives time-series averages of quarterly moments. Low (High) productivity firms are firms within the lowest (highest) quintile of productivity measured by gross output per worker.

Literature

Our paper connects two strands of literature. The common core between the two, which is also the starting point of our study, is the idea that diminishing returns in production at the firm level coupled with heterogeneity in productivity (or span of control) is the dominant force that determines a non-degenerate firm-size distribution. This idea goes back at least to Lucas (1978).

The first strand is the large literature on equilibrium models of single-product firm dynamics with competitive labor markets. Classic examples are Hopenhayn (1992), Hopenhayn and Rogerson (1993), and Luttmer (2011).[5] Recent examples, with applications to the Great Recession, are Clementi and Palazzo (2010) and Arellano, Bai, and Kehoe (2016).

Like these models, our framework can generate empirical distributions of firm size and age. Unlike these models, we microfound the adjustment frictions that firms face through the costly hiring of workers, and thus these costs become equilibrium objects. For example, a key determinant of this cost in our model is the distribution of employees across firm size and firm productivity (and hence across marginal surpluses) since it determines the probability of success in poaching for any given hiring firm. In turn,

[5] For a review of the literature see also Luttmer (2010).

		Response	Decomposition(%)
A. Gross Flows			
Hires			
Poaching	EE_{hir}	2.4	44
Non-employment	NE	0.8	14
Separations			
Poaching	EE_{sep}	-1.5	-28
Non-employment	EN	-0.8	-14
B. Net flows			
Poaching	$EE_{hir} - EE_{sep}$	3.9	72
Non-employment	$NE - EN$	1.5	28
C. Total net flows			
Poaching + Non-employment		5.4	100

Table 3: Decomposition of the net employment response to a productivity shock

Notes: This table presents a decomposition of firm employment growth following a persistent productivity shock using preliminary results from Bagger, Fontaine, Galenianos, and Trapeznikova (2018). Using quarterly Danish matched employer-employee and a statistical model $(ARMA)$ of value added, the authors identify persistent and transitory shocks to value added. They then compute the elasticity of each type of flow rate with respect to a one-standard deviation shock to permanent productivity. The first column reports the response of the given flow to a one standard deviation shock to permanent productivity. For example, following a one standard deviation shock to permanent productivity in quarter t, on average, a firm experiences net employment growth of 5.4 percent. Column two decomposes net employment growth into its components. For example, 44 percent of the increase in employment is due to poaching, while separations due to poaching fall sharply.

this distribution is an equilibrium object sensitive to model parameters and policies.

The second literature comprises a number of papers that model multi-worker firms in frictional labor market. Here, two parallel approaches have been taken: directed search and random search.

The directed search models of Kaas and Kircher (2015) and Schaal (2017) have been able to generate firm dynamics resembling those in the micro data. However, the former considers only search from unemployment. The latter includes on-the-job search, but a key feature of the equilibrium of the directed search framework is worker indifference over firm types. This means that realized worker flows are undirected across firm types, thus this class of models cannot speak to a key feature of the worker flow data on net poaching, i.e. large and low productivity firms systematically lose workers to small and high productivity firms (Haltiwanger, Hyatt, and McEntarfer, 2018; Haltiwanger, Hyatt, Kahn, and McEntarfer, 2018) (Table 2). It is worth remarking, though, that these two papers had different objectives to ours. Schaal (2017) designed a model to study uncertainty shocks in a tractable model of unemployment. Kaas and Kircher (2015) proved that a key advantage of directed search, the efficiency and block-recursivity properties of equilibrium, extends to models with 'large' firms.

In the random search strand, Elsby and Michaels (2013) and Acemoglu and Hawkins (2014) solve models where firms face decreasing returns in production, linear vacancy costs, and wages determined by Nash bargaining. In particular, Elsby and Michaels (2013) find that firm size and productivity heterogeneity decouples the response of the value of the average worker and marginal hire following aggregate productivity shocks, addressing the issues identified in Shimer (2005). Gavazza, Mongey, and Violante (2018) show that a random-matching model of firm dynamics of this type augmented with permanent heterogeneity in returns to scale, as well as financial constraints and convex hiring costs, can accurately replicate the empirical age, size and growth rate distribution of firms. All of these models, again, abstract from search on-the-job.

Finally, we note that random search models with wage posting following Burdett and Mortensen (1998), and its generalization to aggregate shocks in Moscarini and Postel-Vinay (2016) and Coles and Mortensen (2016), do have a firm-size distribution despite constant returns to scale. However, this distribution is entirely generated by the existence of search frictions in the labor market. As frictions disappear, all workers become employed at the most productive firm. One advantage of our framework is that we can decompose how much of size dispersion is due to technology and how much is due to frictions.

Within the random search literature, we build on the sequential auction model of Postel-Vinay and Robin (2002) which has become a workhorse of the literature. Our contribution is to show that, by generalizing this protocol to multi-worker firms, one can maintain a great deal of tractability and solve the model's equilibrium through the distribution of marginal surpluses.

Outline

We first layout the model in a series of steps. In Section 2 we establish the environment and our key assumptions on how agents share value following various stochastic events. Before proceeding further, Section 3 applies these assumptions in a simplified, static framework in order to establish how these assumptions deliver tractability. Section 4 then returns to the fully dynamic model, applying these assumptions in light of the static model and relegating additional details to Appendix A. After defining an equilibrium, Section 5 describes the dynamics of firms in the economy. In Section 6 we calibrate the model to data on net job flows as is customary in the firm dynamics literature, and then compare the model's implications for different types of gross flows to recent data from Haltiwanger, Hyatt, Kahn, and McEntarfer (2018) and Bagger, Fontaine, Galenianos, and Trapeznikova (2018). Section 7 presents our main quantitative exercises. Section 8 concludes.

2 Model environment

In this short section we describe the characteristics of the agents in the economy, how meetings take place in the labor market, and the timing of events. We layout out our key assumptions regarding how workers share value, which we then apply in subsequent sections.

2.1 Environment

Time is continuous and there are no aggregate shocks. The economy is populated by a unit mass of ex-ante identical, infinitely-lived workers and endogenous mass $\mathtt{m}$ of heterogeneous firms.

Workers. Workers are risk neutral and discount the future at rate $\rho > 0$. They inelastically supply one unit of labor at each point in time from one of two states: unemployment or employment. Unemployed workers produce and consume b of the final good, while employed workers consume their wage w.

From both states, individuals search for jobs. Denote by λ^U the meeting rate of unemployed workers, and by $\lambda^E = \xi\lambda^U$ that of employed workers, where ξ denotes an exogenously given relative search intensity of employed workers. In addition to endogenous separations, existing matches break up at exogenous rate δ.

Firms. Firms are characterized by an exogenous, time-varying, idiosyncratic productivity $z \in Z$. They produce a multi-purpose final good using only labor according to the production function $y(z, n)$.[6] For now we assume that n is an integer, and later take the limit as the size of a worker approaches zero. The final good is exchanged in a competitive market and we normalize its price to one.

To hire workers, a firm posts vacancies v subject to cost $c\,(v; z, n)$, with c increasing and convex in v. If the firm exits its scrap value is ϑ.

As well as incumbent firms of measure $\mathtt{m}$, there is an infinite supply of potential entrants. A potential entrant first pays an entry cost c_0, then draws a productivity level z from an initial distribution $\Pi_0\,(z)$ and decides whether to enter or not. A potential entrant that enters becomes an incumbent with n_0 workers that are hired outside of the matching process.

Matching. Workers meet with vacancies in a labor market characterized by search frictions. Search is random. We assume that the total number of meetings between workers and firms is given by a CRS aggregate matching technology $m(\mathtt{s}, \mathtt{v})$ which takes as inputs aggregate vacancies $\mathtt{v}$, and the efficiency

[6] The production function y is increasing and concave in n, with $y(z, 0) = 0$. In addition, we assume that for any z we assume that the Inada conditions hold with respect to n.

units of searching workers $\mathtt{s} = \mathtt{u} + \xi(\bar{\mathtt{n}} - \mathtt{u})$, where $\mathtt{u}$ is the mass of unemployed workers. Given the constant returns to scale matching function, an unemployed worker matches with a vacancy at rate $f(\theta)$, and an employed worker matches with a vacancy at rate $\xi f(\theta)$, where labor market tightness $\theta = \mathtt{v}/\mathtt{s}$. A vacancy matches with an unemployed worker at rate $\phi q(\theta)$ and with an unemployed worker at rate $(1-\phi)q(\theta)$ where $\phi = \mathtt{u}/\mathtt{s}$ and $f(\theta) = q(\theta)/\theta$.

States. Let x be the vector of state-variables for the firm. This includes all state variables of all workers at the firm. For now, we do not specify exactly what is in x and, along the way, define a number of functions that map x at instant t into a new state vector at $t + dt$. The vector x is common knowledge among all workers and the firm. Let i be an indicator function (possibly also a vector) that selects the particular entries of x that identify the worker within a firm (i.e., i is the unique identity of a worker in that firm x).[7] Let $H(x)$ be the distribution of x in the economy, $v(x)$ the number of vacancies created by a firm with state x, and $n(x)$ the size of firm x. The total mass of vacancies and employed workers in the economy are

$$\mathtt{v} = \mathtt{m}\int v(x)\,dH(x) \quad , \quad \mathtt{n} = \bar{\mathtt{n}} - \mathtt{u} = \mathtt{m}\int n(x)\,dH(x).$$

Other distributions that will show up in firm and worker problems are the distribution of vacancies, and the distribution of employment:

$$f(x) = \frac{v(x)}{\mathtt{v}}\mathtt{m}h(x) \quad , \quad g(x) = \frac{n(x)}{\mathtt{n}}\mathtt{m}h(x).$$

Timing. We separate the within-dt timing of events in the model into two parts.

First, all events up to the opening of the labor market are described in Figure 1. Prior to the opening of the labor market, a firm's productivity z is first realized. Following this, workers choose whether to quit the firm, are fired, or employment contracts are renegotiated. Following this, the firm decides whether to stay in operation or exit. A firm that operates produces $y(z, n)$, pays wages according to contracts with its workers, and posts vacancies in the labor market.

Second, the mutually exclusive and collectively exhaustive labor market events that may occur to worker or firm are described in Figure 2. The first branch in Figure 2 describes events that may occur to an unemployed worker. The second and third branch distinguish between events that may affect the value of an incumbent worker i in terms of whether they are an event directly involving the worker,

[7] For example, x is a complete description of IBM and of all its workers. For example, it might contain IBM productivity z, its size n, and all those features of the contracts of the current employees that are needed to forecast IBM's value and the value of each of its workers. The state (x, i) should be read: Here is IBM, characterized by x, and we are assessing the characteristics of the worker named i within IBM.

or an *indirect* event in which either of these occurs to a co-worker $j \neq i$. *Direct* events are a meeting with another firm, or the destruction of the worker's job. *Indirect* events are a co-worker's meeting with another firm, or the destruction of a co-worker's job. Following either of these events, the state x changes, potentially changes the value of the match to worker i. The final branch describes events that impact the firm. The firm may meet an employed or unemployed worker, and following negotiation with the worker and incumbent work-force, emerge either with a new hire or not and new allocation of values to its workers, reflected in updates to the state x. We describe assumptions that limit behavior at each of these nodes through the following assumptions.

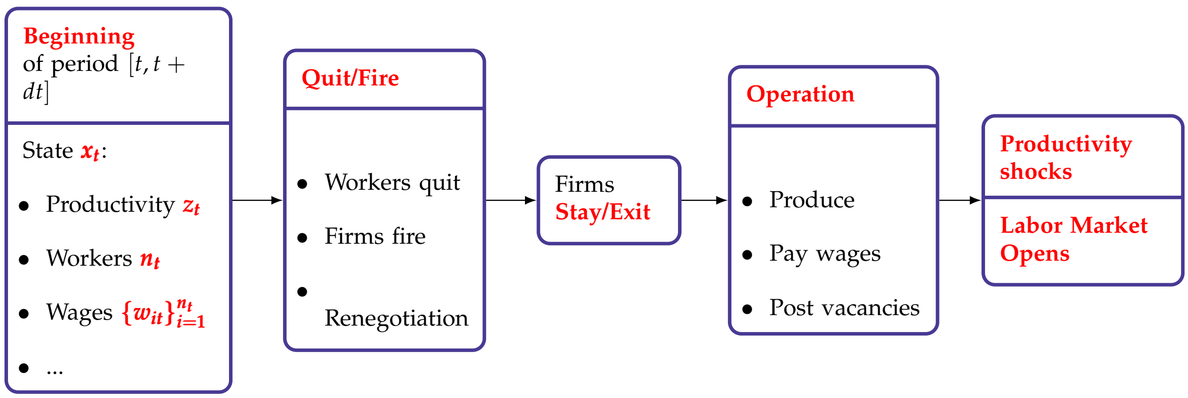


Figure 1: Timing of events prior to the labor market

2.2 Assumptions

The following are a sufficient set of assumptions required to derive all of our main results. It is useful to begin from the definition of a contract.

A contract is binding agreement between the firm and each one of its individual workers which satisfies the following four assumptions:

(A-LC) **Limited commitment.** All parties are subject to limited commitment. In particular,

(a) **Layoffs -** Firms can fire workers at will.

(b) **Hires -** When firms post vacancies, they cannot pre-commit to hire in the case that they meet a worker.

(c) **Wages -** Wages (i.e. firm payments in exchange for worker's labor services) can be changed by the firm at will.

(d) **Quits -** Workers can always quit into unemployment or to another firm when they meet one.

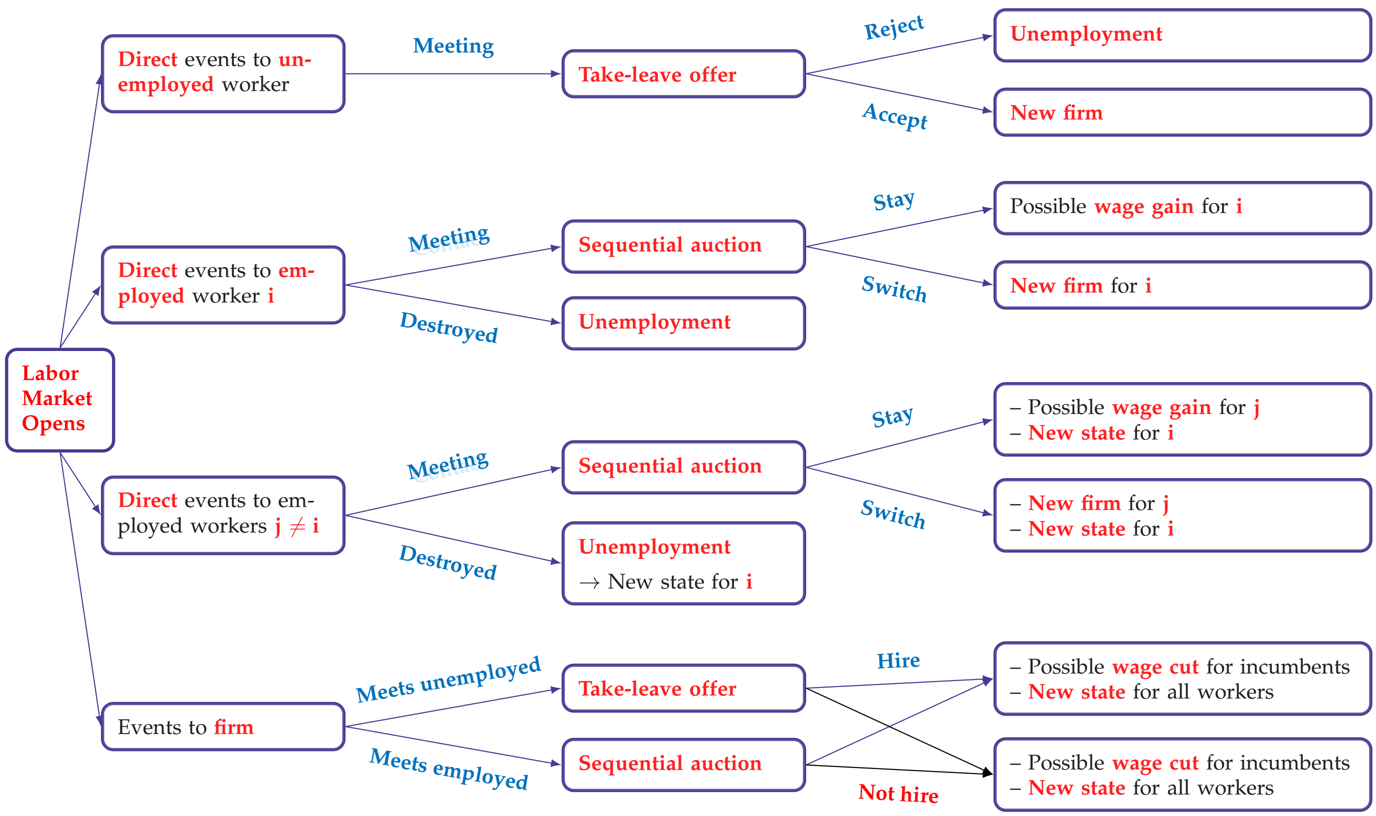


Figure 2: Labor market: Set of mutually exclusive possible labor market events

(e) **Collective agreements –** Workers cannot commit to any other worker inside or outside the firm.

(A-MC) **Mutual consent.** The wage can be modified only by mutual consent, i.e. only if one party can credibly threaten to dissolve the match (the firm by firing, the worker by quitting). A threat is a credible threat when one of the two parties has an outside option that provides a value that is higher than value than under the current contract. Note that this assumption restricts A-LC (c).

(A-BP) **Bargaining Protocol -** All offers are costlessly made by firms to workers and are *take-it-or-leave-it*.[8]

External: Upon meeting an unemployed worker, the firm makes the offer and the worker decides. In the case of a meeting with an employed worker, first the poaching firm makes

[8] This is not necessary. Everything that follows may be computed under an assumption of surplus splitting as in Cahuc Postel-Vinay Robin (2006). For presentation, though, the assumption of *take-it-or-leave-it* as in Postel-Vinay Robin (2002), is cleaner.

the offer, second, the poached from firm makes a counteroffer, and third, the worker decides between the two.

Internal: When the firm bargains with multiple incumbent workers, it does so sequentially (the order is irrelevant). Internal negotiations occur before external ones.

(A-TR) **Transfers -** Any transfer of value—positive or negative—is allowed between each employed worker and the firm. Transfers can be made in exchange for an action. Transfers among workers are not allowed.

Finally, as a tie-breaking rule, we assume that (i) when an employed worker is indifferent between staying and quitting, she stays on the job and (ii) when the firm is indifferent between retaining the worker and firing her, it keeps the worker.

3 Static Example

Our main results are that, given the assumptions above, (i) all decisions are determined by maximizing the joint value of the firm and its incumbent workers (and thus decisions are efficient within the coalition), and (ii) this joint value is only a function of productivity z and size n. In this section we develop a static model in which these results also hold. The manner in which we apply the assumptions in order to derive the results in this section are the same as in the dynamic model, albeit more complicated in the latter. However we believe that most of the intuition for these results can be found in this section. We therefore proceed in detail here, while relegating the lengthy details of the following section to Appendix A.

3.1 Setup

We begin by studying a firm that has $n = 1$ worker. The worker is being paid a wage $w_1 \in (b, z)$, where $b = U$ is the value of unemployment, such that the incumbent worker does not have a credible threat to quit into unemployment. We assume that the firm has sunk the cost of a vacancy, and consider the possible outcomes as that vacancy meets another worker. Later we return to the vacancy posting decision.

3.2 Hires from unemployment

Assume the firm's vacancy meets an unemployed worker. Because of **A-BP** (take-leave offers), the wage the firm offers to the new hire is $w_2 = b$.

Four different cases can arise from the combination of hiring/not hiring and renegotiating/not renegotiating the wage with the incumbent. First we describe the firm's profits in each case. We denote the profit of the firm by the function J, where the first entry gives the firm productivity, the second entry the number of workers, the third the wage paid to worker one, and the fourth the wage paid to worker two if employed.

No hire If the firm does not hire and does not renegotiate with its incumbent, its current profits remain

$$J(z, 1, w_1, \cdot) = y(z, 1) - w_1.$$

If the firm does not hire and does renegotiate with its incumbent, its current profits become

$$J(z, 1, w_1^*, \cdot) = y(z, 1) - w_1^*,$$

where w_1^* is the wage after the internal negotiation between firm and incumbent. Under **(A-MC)**, renegotiation implies that the firm had a credible threat to fire the worker. Under **(A-BP)**, the wage offered will be $w_1^* = b$.

Hire If the firm hires the worker and there is no renegotiation with the incumbent, then profits are:

$$J(z, 2, w_1, b) = y(z, 2) - w_1 - b.$$

If the firm hires the worker and there is renegotiation with the incumbent, profits are:

$$J(z, 2, w_1^*, b) = y(z, 2) - w_1^* - b,$$

where the same logic as above applies to the renegotiation under **(A-MC)** and **(A-BP)**.

We now derive conditions under which these four cases arise.

Hire. A hire *without* renegotiation occurs when the following two conditions hold:

$$\begin{aligned} J(z, 2, w_1, b) &> J(z, 1, \cdot, b) \\ J(z, 2, w_1, b) &> J(z, 1, w_1, \cdot) \end{aligned}$$

The first condition implies no renegotiation by establishing that the threat to fire worker 1 is not credible. Keeping worker one employed at wage w_1 and employing the unemployed worker at wage b delivers a higher value than the threat: firing worker one and hiring the unemployed worker. The second condition is necessary for hiring to be optimal with no renegotiation. Since $J(z,1,\cdot,b) > J(z,1,w_1,\cdot)$, this case arises if and only if

$$J(z,2,w_1,b) > J(z,1,\cdot,b).$$

Since $J(z,2,b,b) > J(z,2,w_1,b)$ is always true, we can arbitrarily extend this to the condition

$$J(z,2,b,b) > J(z,2,w_1,b) > J(z,1,\cdot,b). \tag{1}$$

A hire *with* renegotiation occurs when the following two conditions hold:

$$\begin{aligned} J(z,2,w_1,b) &< J(z,1,\cdot,b) \\ J(z,2,b,b) &> J(z,1,b,\cdot) \end{aligned}$$

The first condition implies renegotiation by establishing that the threat to fire worker 1 is credible. Keeping worker one employed at wage w_1 and employing the unemployed worker at wage b delivers a lower value than the threat: firing worker one and hiring the unemployed worker. The second condition is necessary for hiring to be optimal under the renegotiated wage of b to worker 1 (made as a take-it-or-leave-it offer and accepted by worker 1). This case arises if and only if

$$J(z,2,b,b) > J(z,1,\cdot,b) > J(z,2,w_1,b). \tag{2}$$

We can combine (1) and (2). Note that the order of profits under $(\cdot,b)$ and (w_1,b) only affect whether renegotiation takes place. A necessary and sufficient condition for hiring—irrespective of renegotiation—is that

$$J(z,2,b,b) > J(z,1,\cdot,b). \tag{3}$$

Notice that (3) can be written

$$\begin{aligned} J(z,2,\cdot,\cdot) - b - b &> J(z,1,\cdot,\cdot) - b, \\ J(z,2,\cdot,\cdot) - J(z,1,\cdot,\cdot) &> b. \end{aligned} \tag{4}$$

Now, let $V_i(z,2,w_1,w_2) = w_i$ with $i \in \{1,2\}$ be the value of worker i employed at firm (z,n,w_1,w_2)

and similarly for a firm with $n = 1$. Combining worker and firm values, let $\Omega(z,2) = J(z,2,w_1,w_2) + \sum_{i=1}^{2} V_i(z,2,w_1,w_2)$ be the *joint value* of the firm and its workers, and similarly for $n = 1$. Note that because wages enter linearly and additively $\Omega(z,n) = J(z,n,\cdot,\ldots,\cdot)$, and so $\Omega(z,n)$ does not depend on the wage distribution inside the firm.[9]

Using the definition of joint value, equation (4) can be written as

$$\Omega(z,2) - \Omega(z,1) > U. \tag{5}$$

The decision of hiring from unemployment does not depend on wages, but only on productivity, size, and the value of unemployment U.

We define one additional object, the *total surplus* $S(z,n)$. This is a natural extension of the familiar function from one worker firm models of the labor market: $S(z,n) = \Omega(z,n) - nU$. Our final Bellman equation for the firm in the dynamic model will be written in terms of this function. Using this definition the firm hires if and only if the marginal surplus is positive:

$$\frac{S(z,2) - S(z,1)}{2-1} > 0$$

No hire. For completeness, consider the cases where no hiring occurs. No hire, but with renegotiation happens when:

$$\begin{aligned} J(z,1,\cdot,b) &> J(z,1,w_1,\cdot) \\ J(z,1,b,\cdot) &\geq J(z,2,b,b). \end{aligned}$$

The first condition guarentees renegotiation since the firm has a credible threat to fire the worker and replace them with the unemployed worker. The second condition is required for no hiring to take place. Since the first condition always holds, the wage of an incumbent worker is always negotiated down when no hiring occurs. That is, the case of no hire, no renegotiation never occurs. Finally, note that the joint value is unaffected by renegotiation, so in case of no hire, the joint value remains $\Omega(z,1) = z$.[10]

[9] Two relevant cases that would violate this condition are (i) if worker's effort enters the production function and effort depends on the wage and (ii) concave utility.

[10] The value before the hire was $\Omega(z,1) = J(z,1,w_1) + V_1(z,1,w_1) = z - w_1 + w_1 = z$. The joint value after the hire is $\Omega(z,1) = J(z,1,b) + V_1(z,1,b) = z - b + b = z$

3.3 Hires from employment

Now suppose that the worker matched with the firm's vacancy is currently employed at another firm with productivity z', and also with $n' = 1$. According to **A-BP**, the potential hiring firm makes the first offer. Denote this wage offer $w_2^* = \bar{w}$. Again, four different cases can arise, a combination of hiring/not hiring and renegotiating/not renegotiating the wage with the incumbent.

A hire *without* renegotiation occurs when:

$$\begin{aligned} J(z, 2, w_1, \bar{w}) &> J(z, 1, \cdot, \bar{w}) \\ J(z, 2, w_1, \bar{w}) &> J(z, 1, w_1, \cdot) \end{aligned}$$

No renegotiation is guarenteed by the first condition, which states that the threat to fire worker one and replace him with the poached worker is not credible. The firm would prefer to hire the new worker at $\bar{w}$ and keep the existing worker at w_1 than to hire the new worker at $\bar{w}$ and fire the existing worker. Given no renegotiation of w_1, the second condition implies that hiring is optimal. Thus, this case arises if and only if

$$J(z, 2, w_1, \bar{w}) > \max \{J(z, 1, \cdot, \bar{w}), J(z, 1, w_1, \cdot)\}. \tag{6}$$

A hire *with* renegotiation occurs when:

$$\begin{aligned} J(z, 2, w_1, \bar{w}) &< J(z, 1, \cdot, \bar{w}) \\ J(z, 2, b, \bar{w}) &> J(z, 1, b, \cdot) \end{aligned}$$

Renegotiation is guarenteed by the first condition, which states that the threat to fire the incumbent is credible. Under take-it-or-leave-it offers, the renegotiated wage will be b, and so the second condition implies that hiring is optimal.[11] Since $J(z, 1, b, \cdot) > J(z, 1, \cdot, \bar{w})$, this case arises if and only if

$$J(z, 2, b, \bar{w}) > J(z, 1, b, \cdot) > J(z, 1, \cdot, \bar{w}) > J(z, 2, w_1, \bar{w}). \tag{7}$$

We can combine (6) and (7). Hiring occurs if and only if

$$J(z, 2, b, \bar{w}) > J(z, 1, b, \cdot). \tag{8}$$

[11] Recall that **A-BP** requires that internal renegotiations occur first, and that **A-LC** implies that even once the incumbent's wage has been renegotiated, the firm is not committed to hiring. Hence this is the correct inequality. As opposed to $J(z, 1, b, \bar{w}) > J(z, 1, w_1, \cdot)$. One may usefully think of the two conditions sequentially. *First*, the firm uses the threat of renegotiation to push down the incumbent's wage. *Second*, the firm takes this lower wage of the incumbent as given and decides whether to hire or not.

To show this, we start by showing sufficiency. Suppose that $J(z, 2, b, \varpi) > J(z, 1, b, \cdot)$ holds. There are then two possible cases. First, $J(z, 1, \cdot, \varpi) > J(z, 2, w_1, \varpi)$. Then, by construction, we are in the hire with renegotiation case because the middle inequality is always satisfied. Hiring occurs. Second, $J(z, 2, w_1, \varpi) > J(z, 1, \cdot, \varpi)$. Noting that $J(z, 2, b, \varpi) > J(z, 1, b, \cdot)$ implies that $J(z, 2, w_1, \varpi) > J(z, 1, w_1, \cdot)$, then we are in the hire without renegotiation case and hiring occurs.

Now consider the necessary part of the statement. In the hire with renegotiation case, the implication is obvious. In the hire without renegotiation case, take $J(z, 2, w_1, \varpi) > \max\{J(z, 1, \cdot, \varpi), J(z, 1, w_1, \cdot)\}$, add b to each side and subtract w_1 to get $J(z, 2, b, \varpi) > \max\{J(z, 2, b, \varpi) - w_1, J(z, 1, b, \cdot)\}$, which implies $J(z, 2, b, \varpi) > J(z, 1, b, \cdot)$.

Notice that (8) can be rewritten as

$$
\begin{aligned}
J(z, 2, \cdot, \cdot) - b - \varpi &> J(z, 1, \cdot, \cdot) - b, && (9)\\
\Omega(z, 2) - \Omega(z, 1) &> \varpi. && (10)
\end{aligned}
$$

The decision of hiring from unemployment does not depend on incumbent wages, but only on productivity, size, and the wage that the firm offers to the poached employed worker.

We now determine ϖ. The poached firm is willing to pay up to ϖ that satisfies $J(z', 1, \varpi, \cdot) = J(z', 0, \cdot, \cdot) = 0$. This implies that $\varpi = y(z', 1) = z'$. Therefore, if $\Omega(z, 2) - \Omega(z, 1) > z'$, then the poaching firm will offer exactly $\varpi = z'$. And if $\Omega(z, 2) - \Omega(z, 1) < z'$, then the poaching firm will offer $\varpi = \Omega(z, 2) - \Omega(z, 1)$, which will be matched by the poached firm. Finally, note that $\Omega(z', 1) - \Omega(z', 0) = z'$. Using this, we observe that hiring occurs if and only if

$$
\Omega(z, 2) - \Omega(z, 1) > \Omega(z', 1) - \Omega(z', 0). \tag{11}
$$

This condition is written entirely in terms of joint values.

Finally, using the notation of the total surplus, note that this can be written as

$$
\frac{S(z, 2) - S(z, 1)}{2 - 1} > \frac{S(z', 1) - S(z', 0)}{1 - 0}.
$$

That is, poaching occurs if and only iff the marginal surplus of the worker is larger at the poaching firm.[12]

[12] The case when the firm meets worker 1 at a firm with $(z', 2, w_1, w_2)$ is similar. In this case the poached-from firm will be able to reduce w_2, but this is inconsequential for the argument. Suppose the poached-from firm renegotiates w_2 to w_2^*, under **(A-LC)** the firm would still be prepared to pay up to ϖ that satisfies $J(z', 2, \varpi, w_2^*) = J(z', 1, \cdot, w_2^*)$. Rearranging this we have $\varpi = J(z', 2, \cdot, \cdot) - J(z', 1, \cdot, \cdot) = \Omega(z', 2) - \Omega(z', 1)$. A hire therefore occurs if $\Omega(z, 2) - \Omega(z, 1) > \Omega(z', 2) - \Omega(z', 1)$.

3.4 Vacancy Posting

To simplify the exposition, suppose firms only meet unemployed workers. Let v be the number of vacancies posted and $c(v)$ the associated cost. Recall that the rate at which the vacancy will meet one worker is then qv. The firm's initial state is $(z, 1, w_1, \cdot)$. Let $\bar{J}^f(z, 1, w_1, \cdot)$ be the value of the firm under its optimal vacancy decision:

$$
\begin{aligned}
\bar{J}^f(z,1,w_1,\cdot) &= \max_v \bar{J}(z,1,w_1,\cdot,v) \\
\bar{J}(z,1,w_1,\cdot,v) &= qv \max\left\{J(z,1,w_1^*,\cdot), J(z,2,w_1^*,b)\right\} + (1-qv) J(z,1,w_1,\cdot) - c(v)
\end{aligned}
$$

and let v^f denote the maximand, which satisfies the first order condition

$$
\begin{aligned}
c_v\left(v^f(z,1,w_1,\cdot)\right) &= q \underbrace{\max\left\{J(z,1,w_1^*,\cdot), J(z,2,w_1^*,b)\right\} - J(z,1,w_1,\cdot)}_{R(z,1,w_1,\cdot)}, \\
v^f(z,1,w_1,\cdot) &= c_v^{-1}\Big(qR(z,1,w_1,\cdot)\Big)
\end{aligned}
$$

The state $(z, 1, w_1, \cdot)$ can be such that three cases arise which determine the return on a vacancy $R(z, 1, w_1, \cdot)$, which in turn pins down v^f:

$$
R(z,1,w_1,\cdot) = \begin{cases} J(z,2,w_1,b) - J(z,1,w_1,\cdot) & , \quad \textbf{Case 1} \text{ - Hire without renegotiation with the incumbent} \\ J(z,2,b,b) - J(z,1,w_1,\cdot) & , \quad \textbf{Case 2} \text{ - Hire with renegotiation with the incumbent} \\ J(z,2,b,\cdot) - J(z,1,w_1,\cdot) & , \quad \textbf{Case 3} \text{ - No hire with renegotiation with the incumbent} \end{cases}
$$

Case 1. Note that in this case the outcome is also efficient. The worker's value does not decrease, and by the fact that a hire occurs, the firm's value must increase. Furthermore, note that $R(z_1, w_1, \cdot) = \Omega(z, 2) - \Omega(z, 1) - U$, which we use below.

Inefficient cases. In Cases 2 and 3, in addition to the possibility of hiring so as to increase output, the firm gains from posting vacancies by attaining a credible threat to cut the incumbents wage. From the perspective of the joint value—including vacancy costs—this behavior is inefficient. Costly vacancies are using to transfer value between worker and firm. Ex-ante the incumbent worker realizes this incentive to over post vacancies and, under **(A-TR)**, can make a transfer to the firm in exchange for a level of vacancy posting that is preferred by the worker. The optimal transfer scheme is naturally obtained by having the worker solve a simple mechanism design problem.

Case 3. Suppose that w_1 is such that Case 3 would obtain. The worker solves the following for the worker-optimal vacancies v^w and transfer t:

$$\begin{aligned}
&\max_{v^w,\, t\geq 0} qv^w b + (1 - qv^w)\, w_1 - t \qquad (12)\\
&s.t.\\
&\bar{J}(z, 1, w_1, \cdot, v^w) + t \geq \bar{J}(z, 1, w_1, \cdot, v^f(z, 1, w_1, \cdot)) \quad (IC)
\end{aligned}$$

The objective function takes into the account that the worker's wage will still be negotiated down following a meeting, but seeks a lower level of vacancies which reduces the probability of this occuring. The (IC) constraint implies that the firm will prefer to choose $v^w < v^f$ and receive the transfer than to choose v^f. Because of the linearity of the objective function, (IC) holds with equality. We can then substitute out t to obtain

$$\max_{v^w} qv^w b + (1 - qv^w)\, w_1 + \left[\bar{J}(z, 1, w_1, \cdot, v^w) - \bar{J}(z, 1, w_1, \cdot, v^f(z, 1, w_1, \cdot))\right]. \qquad (13)$$

Since $\bar{J}(z, 1, w_1, \cdot, v^f(z, 1, w_1, \cdot))$ is independent of v^w, it drops out of the optimization problem. We can then substitute in the expressions for $\bar{J}$ under Case 3, where because the worker is not hired

$$\bar{J}(z, 1, w_1, \cdot, v^w) \quad = \quad qv^w J(z, 1, b, \cdot) + (1 - qv^w) J(z, 1, w_1, \cdot) - c(v^w).$$

Substituting this in and collecting terms—noting that $b = U$ and $w_1 = V(z, 1, w_1, \cdot)$—gives

$$\max_{v^w} qv^w\, [J(z, 1, b, \cdot) + U] + (1 - qv^w)\, [J(z, 1, w_1, \cdot) + V(z, 1, w_1, \cdot)] - c(v^w).$$

Now note that both terms in square brackets represent the same total value $\Omega(z, 1)$, but distributed differently across worker and firm. The return to search is always $\Omega(z, 1)$, independently of the outcome of search, so no vacancy is posted. In other words, the optimal transfer is the one that restores efficiency. Intuitively, excess vacancy posting redistributes value from worker to firm inefficiently because it entails wasteful costs (i.e. this is just an application of the Coase theorem). We conclude that, once again, in this case, the joint value is sufficient to characterize the vacancy decision.

Case 2. Suppose now parameters are such that we are in Case 2, then the optimal transfer scheme solves (13) as before. We can then substitute in the expressions for $\bar{J}$ under Case 3, where because the worker is hired

$$\bar{J}(z,1,w_1,\cdot,v^w) \quad = \quad qv^w J(z,1,b,b) + (1-qv^w)J(z,1,w_1,\cdot) - c(v^w).$$

Substituting this into (13) and proceeding as before we have

$$\max_{v^w} \; qv^w \underbrace{[J(z,1,b,b)+U]}_{\Omega(z,2)-U} + (1-qv^w)\underbrace{[J(z,1,w_1,\cdot)+V(z,1,w_1,\cdot)]}_{\Omega(z,1)} - c(v^w).$$

Dropping terms that are independent of v^w, the optimal vacancy solves

$$\max_{v^w} \; qv^w[\Omega(z,2) - \Omega(z,1) - U] - c(v^w).$$

This establishes, again, that the transfer restores efficiency in vacancy posting, with the return being equal to the total change in firm value net of the cost of the new hire U. Under our earlier definition of firm total surplus $S(z,n) = \Omega(z,n) - nU$, it becomes clear that the optimal vacancy equates the marginal surplus to the effective marginal cost $q^{-1}c_v(v)$.

$$\max_{v^w} \; qv^w \underbrace{\left[\frac{S(z,2)-S(z,1)}{2-1}\right]}_{\text{Marginal surplus}} - c(v^w).$$

Combined Note that in both Case 1 and Case 3, the optimal vacancy decision is characterized by

$$v = \arg\max_{v} \; \Omega(z,2) - \Omega(z,1) - U.$$

Therefore, combining all three cases, we can state the efficient vacancy posting decision in general as:

$$v = \arg\max_{v} \; qv\left(\max\left\{\Omega(z,2) - \Omega(z,1) - U, 0\right\}\right) - c(v).$$

This establishes that the vacancy decision is independent of w_1. In terms of surplus note that

$$v = \arg\max_{v} \; qv\left(\max\left\{\frac{S(z,2)-S(z,1)}{2-1}, 0\right\}\right) - c(v).$$

Dynamic model In our full dynamic model a firm will obviously employ more than one worker. However, it is then easy to show (see Appendix) that the efficient transfer scheme can be implemented by a single worker who makes a transfer to the firm (which leaves the firm indifferent) conditional of the following action: (i) the firm posts the efficient number of vacancies and (ii) the firm requires in exchange a transfer from all other workers that leaves them indifferent. Under such a scheme, the initiating worker is strictly better off.

3.5 Exit

We now consider the exit decision of a firm under our assumptions, again starting with a firm with state $(z, 1, w_1, \cdot)$. Recall that the value of a firm upon exit is $c_f > 0$. Let $\mathbf{J}(z, 1, w_1, \cdot)$ be the pre-exit-decision value of the firm:

$$\mathbf{J}(z, 1, w_1) = \max\left\{J\left(z, 1, w_1^*\right), \vartheta\right\}$$

where w_1^* is a possibly renegotiated wage contingent on the firm remaining in operation. We have three cases. If

$$J\left(z, 1, w_1\right) - c_f > 0$$

then the firm continues operating and has no credible threat to reduce the incumbent wage. If

$$J\left(z, 1, w_1\right) - c_f < 0 < J\left(z, 1, b\right) - c_f$$

then the firm has a credible threat to cut the wage to b (threat of exit) and continues operating. Finally, if

$$J\left(z, 1, b\right) - c_f < 0$$

the firm exits. Note that $J\left(z, 1, b\right) = \Omega\left(z, 1\right) - 1 \times U$. Thus, more generally, exit only happens when

$$\Omega\left(z, n\right) - nU < c_f.$$

or $S(z, n) < c_f$. And, once again, the decision can be fully characterized in terms of the joint surplus.

3.6 Quits and Layoffs

To describe the quits and layoffs decisions it is useful to consider a firm with $n = 2$ workers denoted by $i = 1, 2$. Let's start with layoffs. Assume $w_1 < z$. The firm has a credible threat to fire worker 2 whenever

$$J(z, 1, w_1, \cdot) \quad > \quad J(z, 2, w_1, w_2),$$

in which case worker 2 could be forced to take a wage cut from w_2 to b. If at the renegotiated wage $w_2 = b$ we still have

$$J(z,1,w_1,\cdot) > J(z,2,w_1,b)$$

then there is a layoff. In terms of joint values, the above condition is written

$$\Omega\,(z,2) - \Omega\,(z,1) < U,$$

or in terms of joint surplus

$$\frac{S(z,2) - S(z,1)}{2-1} < 0$$

With respect to quits, the threat of quit to unemployment prevents the firm from paying wages below $U = b$.[13]

3.7 Internal renegotiation with multiple workers: the case of hires from unemployment

In this section we illustrate how internal renegotiation occurs in the case the firm employs multiple workers. It is enough to consider the case $n = 2$ and $w_2 > w_1$, without loss of generality. Suppose the firm has posted vacancies under the efficient policy (and thus would want to hire) and has met an unemployed worker. We have three cases to consider.

First, the firm does not have a credible threat to renegotiate with any of its incumbents when:

$$\begin{aligned} J\,(z,3,w_1,w_2,b) &> \max\left\{J\,(z,2,w_1,b)\,, J(z,2,w_2,b)\right\}, \\ J\,(z,3,w_1,w_2,b) &> J\,(z,2,w_1,w_2)\,. \end{aligned}$$

The first condition says that hiring is with current wages is preferred to firing either of the incumbents and hiring. There is no renegotiation. The second condition says that, given this, hiring is optimal. condition can be written, as usual in terms of total values:

$$\begin{aligned} \Omega\,(z,3) - \Omega\,(z,2) &> V_2\,(z,2,b,w_2)\,, \\ \Omega\,(z,3) - \Omega\,(z,2) &> U. \end{aligned}$$

[13]Note that the particular order in which values of workers are reduced is immaterial to the condition $\Omega\,(z,2) - \Omega\,(z,1) < U$. One could lower the wages of both workers, increasing the value of the firm, but a worker must be fired if $J(z,2,w_1,b) < J(z,1,w_1,\cdot)$ for any $w_1 \geq b$. Since one can always add $2b - w_1$ to both sides of the inequality to obtain $J(z,2,b,b) + b < J(z,1,b,\cdot)$, which gives $\Omega\,(z,2) - \Omega\,(z,1) < U$.

Second, the firm may have a credible threat to renegotiate with the higher paid worker 2 ($w_2 > w_1$) only:

$$
\begin{aligned}
J(z,2,w_1,\cdot,b) &> J(z,3,w_1,w_2,b) > J(z,2,\cdot,w_2,b), \\
J(z,3,w_1,b,b) &> J(z,2,w_1,b).
\end{aligned}
$$

The first condition states that the threat is not credible for worker 1, but is for worker 2. The second condition states that the firm is better off hiring after having renegotiated the wage of worker 2. These conditions can be written, as usual in terms of total values, as

$$
\begin{aligned}
V_2(z,2,b,w_2) &> \Omega(z,3) - \Omega(z,2) > V_1(z,2,w_1,b), \\
\Omega(z,3) - \Omega(z,2) &> U.
\end{aligned}
$$

Third, the firm has a credible threat to renegotiate with *both* workers:

$$
\begin{aligned}
\min\left\{J\left(z,2,w_1,b\right), J(z,2,w_2,b)\right\} &> J\left(z,3,w_1,w_2,b\right), \\
J\left(z,3,b,b,b\right) &> J\left(z,2,b,b\right).
\end{aligned}
$$

The first condition states that if either worker was fired and the unemployed worker hired, then the value would increase relative to keeping wages fixed. The second condition guarentees that the firm prefers to hire conditional on renegotiation. Again, these conditions can be written in terms of total values

$$
\begin{aligned}
\min\left\{\, V_1\left(z,2,w_1,b\right)\ ,\ V_1\left(z,2,w_2,b\right)\,\right\} &> \Omega\left(z,3\right) - \Omega\left(z,2\right) \\
\Omega\left(z,3\right) - \Omega\left(z,2\right) &> U
\end{aligned}
$$

Under **(A-BP)**, the firm proceeds sequentially to make take-leave offers to all workers for which it has a credible firing (swapping) threat (in this case, both workers). The first condition above implies that for both workers $i \in \{1,2\}$

$$
w_i > \Omega(z,3) - \Omega(z,2).
$$

First, this implies that the order of the internal negotiation between firm and workers doesn't matter and there is no scope for strategic behavior: because of the take-leave protocol, no worker would ever reject the firm offer and end up unemployed.[14]

[14]The take-leave offer is key to shutting down strategic interactions. Intuitively, a firm uses the one unemployed worker at the door to threat many incumbents sequentially. Recall that workers are not allowed to make transfers between themselves. This is where that assumption is binding.

3.8 Summary

In summary we found the following conditions in terms of joint surplus, which we now write more generally in terms of n:

- The firm exits if and only if the average surplus is less than the outside value of the firm

$$S(z,n) < c_f$$

- The firm fires workers if and only if the marginal surplus is negative

$$\frac{S(z,n+1)-S(z,n)}{(n+1)-(n)} \leq 0 \quad \leftrightarrow \quad S_n(z,n) \leq 0$$

- If and only if the marginal surplus is non-negative, then the firm posts vacancies, v that solve

$$\max_v qv\left(\frac{S(z,n+1)-S(z,n)}{(n+1)-(n)}\right) - c(v) \quad \leftrightarrow \quad S_n(z,n) > S_n(z',n') \max_v qvS_n(z,n) - c(v)$$

- Since the condition for vacancy posting and hiring from unemployment are identical, then the firm always hires when a vacancy meets an unemployed worker.

- If the firm's vacancy meets an employed worker, then it hires the worker if and only if

$$\frac{S(z,(n+1))-S(z,n)}{(n+1)-n} > \frac{S(z',n')-S(z',n'-1)}{n'-(n'-1)} \quad \leftrightarrow \quad S_n(z,n) > S_n(z',n')$$

We now derive similar conditions for the dynamic model with a discrete workforce. Then, taking the size of one worker—here equal to 1—to zero, we naturally express the above conditions in terms of marginal surpluses $S_n(z,n)$, as shown above to the right. Observe that the final condition gives an early suggestion that the relevant aggregate state variable will be the distribution of $S_n(z',n')$ since this determines the dynamics of n within the firm.

4 Dynamic model

We first specify the value function of an individual worker i in a firm with arbitrary state x: $V(x,i)$. We then specify the value function of the firm: $J(x)$. Combining all worker's value functions and that of the firm we define the joint value: $\Omega(x)$. We then apply the assumptions from Section 2.2 which allow us to reduce that state-space of the joint value x to only the number of workers and productivity of the firm (n,z). Finally we take the continuous work force limit to derive a Hamilton-Bellman-Jacobi (HJB) equation for $\Omega(n,z)$ Applying the definition of total surplus used above we arrive at a HJB equation in $S(n,z)$ which we use to construct the equilibrium. Many additional details that expand on the proofs used for the static game are contained in Appendix A.

4.1 Worker value function: V

As in the static example, let U be the value of unemployment. It is convenient to separate the value to worker i of employment at firm x into that before the exit, quit and layoff decisions, $\mathbf{V}(x,i)$, and that after these decisions, $V(x,i)$.

Value of unemployment. Let $h_U(x)$ denote how the state of firm x is updated when it hires an unemployed worker. Let $\mathcal{A}$ denote the set of firms that an unemployed would accept an offer of employment from. The value of unemployment U therefore satisfies

$$\rho U = b + f(\theta) \int_{x \in \mathcal{A}} \left[\mathbf{V}\left(h_U(x), i\right) - U \right] dF(x)$$

where F is the vacancy-weighted distribution of firms. The worker remains unemployed whenever $x \notin \mathcal{A}$.

Value of employment - First stage. To relate the value of the worker pre separation, $\mathbf{V}(x,i)$, to that post separation, $V(x,i)$, we require the following notation regarding firm and co-worker actions that the worker takes as given:

- Let $\epsilon(x) \in \{0,1\}$ denote the exit decision of firm, and $\mathcal{E} = \{x : \epsilon(x) = 1\}$ the set of x's for which the firm exits.
- Let $\ell(x) \in \{0,1\}^{n(x)}$ be a vector of length $n(x)$ of zeros and ones, with generic entry $\ell_i(x)$ that characterizes the firm's decision to lay off incumbent worker i, and $\mathcal{L} = \{(x,i) : \ell_i(x) = 1\}$ the set of (x,i) such that worker (x,i) is laid off.
- Let $q^U(x) \in \{0,1\}^{n(x)}$ be a vector of length $n(x)$, with generic entry $q_i^U(x)$ that characterizes an incumbent workers' decisions to quit, and $\mathcal{Q}^U \left\{(x,i) : q_i^U(x) = 1\right\}$ the set of (x,i) such that worker (x,i) quits into unemployment.

- Let $\kappa(x) = (1 - \ell(x)) \circ (1 - q_U(x))$ be an element-wise product vector that identifies workers that are kept in the firm, and $\mathcal{S} = \mathcal{L} \cup \mathcal{Q}^U = \{(x,i) : \kappa_i(x) = 0\}$, the set of (x,i) for which worker (x,i) will separate into unemployment.

- Let $s(x, \kappa(x))$ denote how the state of firm x is updated when $\kappa(x)$ workers are kept. This includes any renegotiation.

Given these sets and functions, $\mathbf{V}(x,i)$ satisfies:

$$\mathbf{V}(x,i) = \epsilon(x)U + (1-\epsilon(x)) \left[\mathbb{I}_{\{(x,i)\in\mathcal{S}\}} V(s(x,\kappa(x)), i) + \mathbb{I}_{\{(x,i)\notin\mathcal{S}\}} U \right]$$

Value of employment - Second stage. It is helpful to characterize the value of employment post separation decisions, $V(x,i)$, in terms of the three distinct types of events described in Figure 2. First, the value changes due to *'Direct'* labor markets shocks to worker i, $V_D(x,i)$. These include her match being destroyed exogenously or meeting a new potential employer. Second, the value is updated due to labor market shocks hitting other workers in the firm, $V_I(x,i)$, including their matches being exogenously destroyed or them meeting new potential employers. These events have an *'Indirect'* affect on worker i. Third, the value is affected by events on the *'Firm'* side, $V_F(x,i)$, including the firm contacting new workers and receiving productivity shocks. Combining all these and exploiting the fact that in continuous time these are mutually exclusive events, we obtain

$$\rho V(x,i) = w(x,i) + \rho V_D(x,i) + \rho V_I(x,i) + \rho V_F(x,i),$$

where $w(x,i)$ is the wage paid to worker i.

Direct events. We first consider changes in value due to labor market shocks directly to worker i in firm x, $V_D(x,i)$. Exogenous separation shocks arrive at rate δ and draws of outside offers arrive at rate $\xi f(\theta)$ from the vacancy-weighted distribution of firms F. If worker i receives a sufficiently good outside offer x', she quits to the new firm. We denote by $\mathcal{Q}^E(x,i)$ the set of such quit-firms x' for i. Otherwise, the worker remains with the current firm but with an updated contract. Thus:

$$\begin{aligned}\rho V_D(x,i) = \; & \underbrace{\delta [U - V(x,i)]}_{\text{Exogenous separation}} + \xi f(\theta) \underbrace{\int_{x' \in \mathcal{Q}^E(x,i)} \left[\mathbf{V}(h_E(x,i,x'), i) - V(x,i) \right] dF(x')}_{EE \text{ Quit}} \\ & + \xi f(\theta) \underbrace{\int_{x' \notin \mathcal{Q}^E(x,i)} \left[\mathbf{V}(r(x,i,x'), i) - V(x,i) \right] dF(x')}_{\text{Retention}},\end{aligned}$$

where $h_E(x,i,x')$ describes how the state of a poaching firm x' gets updated when it hires worker i from firm x. Similarly, $r(x,i,x')$ describes how x is updated when—after meeting firm x'—worker i in firm x is retained and renegotiates its value. In all functions with three arguments (x,i,x'), the first argument denotes the worker origin firm, the second identifies the worker, and the third the potential destination firm.

Indirect events. We next characterize how the value to worker i in firm x changes in response to the same labor market shocks (exogeneous separations and outside offers) hitting other workers in the firm, the component $V_I(x,i)$:

$$\begin{aligned}\rho V_I(x,i) = \sum_{j\neq i}^{n(x)} \Bigg\{ & \underbrace{\delta \left[\mathbf{V}(d(x,j),i) - V(x,i) \right]}_{\text{Exogenous separation}} + \xi f(\theta) \underbrace{\int_{x'\in\mathcal{Q}^E(x,j)} \left[\mathbf{V}\left(q_E(x,j,x'),i\right) - V(x,i)\right] dF(x')}_{EE\text{ Quit}} \\ & + \xi f(\theta) \underbrace{\int_{x'\notin\mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r(x,j,x'),i\right) - V(x,i)\right] dF(x')}_{\text{Retention}} \Bigg\},\end{aligned}$$

where $d(x,j)$ updates x when worker j exogenously separates, and $q_E(x,j,x')$ when worker j quits to firm x'.

Firm events. Finally, we consider events that directly impact the firm and hence indirectly its workers, the component $V_F(x,i)$ of the worker's value, taking as given the firm's vacancy posting policy $v(x)$ and actions:

$$\begin{aligned}
\rho V_F(x,i) &= \\
UE\text{ Hire} &\quad +\phi q(\theta) v(x) \left[\mathbf{V}(h_U(x),i) - V(x,i)\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\text{ Threat} &\quad +\phi q(\theta) v(x) \left[\mathbf{V}(t_U(x),i) - V(x,i)\right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\text{ Hire} &\quad +(1-\phi) q(\theta) v(x) \int_{x\in\mathcal{Q}^E(x',i')} \left[\mathbf{V}\left(h_E(x',i',x),i\right) - V(x,i)\right] dG(x',i') \\
EE\text{ Threat} &\quad +(1-\phi) q(\theta) v(x) \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(t_E(x',i',x),i\right) - V(x,i)\right] dG(x',i') \\
\text{Shock} &\quad +\Gamma_z[\mathbf{V},V](x,i)
\end{aligned}$$

where $t_U(x)$ describes the evolution of x when an unemployed worker is met and not hires, but and could be possibly used as a threat in firm x. Similarly, $t_E(x',i',x)$ describes the evolution of x when worker i' employed at firm x' is met, not hired, but could be used as a threat. And $G(x',i')$ gives the joint distribution of firms x' and worker types within firms i'.

Finally, $\Gamma_z[\mathbf{V}, V](x,i)$ identifies the contribution of productivity shocks z to the Bellman equation: at this stage we only require that the productivity process is Markovian with an infinitesimal generator. Later we will specialize this to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$ such that[15]

$$\Gamma_z[\mathbf{V}, V](x,i) = \mu(z) \lim_{dz\to 0} \frac{\mathbf{V}((x, z+dz), i) - V(x,i)}{dz} + \frac{\sigma^2(z)}{2} \lim_{dz\to 0} \frac{\mathbf{V}((x, z+dz), i) + \mathbf{V}((x, z-dz), i) - 2V(x,i)}{dz^2} \tag{14}$$

In the case that $\mathbf{V} = V$, this becomes the standard expression for a diffusion with first and second derivatives: $\Gamma_z[\mathbf{V}, V](x,i) = \mu(z)V_z(x,i) + \frac{1}{2}\sigma^2 V_{zz}(x,i)$.[16]

Finally, note that in the event of productivity changes or $n(x)$ changes because of exogenous events, the worker will want to reassess whether to stay with the firm or not. Additionally, the firm may want to reassess whether to exit or fire some workers. Hence we have the bold value $\mathbf{V}$ in any case where the state changes.

4.2 Firm value function: J

Consistent with the notation we used for workers' values, let $\mathbf{J}(x)$ and $J(x)$ be the values of the firm at these two stages within an interval of time dt. For now, we take the vacancy creation decision $v(x)$ as given. At the end of the section we describe the expected value of an entrant firm.

First stage. Consistent with the first stage worker value function, we define the firm value before the exit/layoff/quit decision:

$$\mathbf{J}(x) = \epsilon(x)\vartheta + [1-\epsilon(x)] J(s(x, \kappa(x))),$$

where we recall that ϑ is the scrap value on exit to the firm.

Second stage. Let $J(x)$ be the value of a firm with state x *after* the layoff/quit, exit and given a vacancy policy $v(x)$. It is convenient to split the value of the firm, as we did for the value of the worker, into

[15]Note that in (14) we abuse notation and write the state as (x, z) with some redundancy since z is clearly a member of x.

[16]Note that we are not constrained to a diffusion process. We could also consider a Poisson process where, at exogenous rate η, z jumps according to the transition density $\Pi(z, z')$:

$$\Gamma_z[\mathbf{V}, V](x,i) = \eta \left[\sum_{z'\in Z} \mathbf{V}((x, z'), i)\, \Pi(z', z) - V(x,i) \right]$$

three components

$$\rho J\left(x\right)=\underbrace{y\left(x\right)-\sum_{i=1}^{n(x)}w_{i}\left(x,i\right)}_{\text{Flow profits}}\quad+\quad\underbrace{\rho J_{W}\left(x\right)}_{\text{Workforce events}}\quad+\quad\underbrace{\rho J_{F}\left(x\right)-c\left(v\left(x\right),x\right)}_{\text{Firm events net of vacancy costs}}\ .$$

The component $J_W\left(x\right)$ is given by:

$$\begin{aligned}
\rho J_{W}\left(x\right) &= \\
\text{Destruction} &\quad \delta\sum_{i=1}^{n(x)}\left[\mathbf{J}\left(d(x,i)\right)-J\left(x\right)\right] \\
\text{Retention} &\quad +\ \xi f(\theta)\sum_{i=1}^{n(x)}\int_{x'\notin\mathcal{Q}^{E}(x,i)}\left[\mathbf{J}\left(r\left(x,i,x'\right)\right)-J\left(x\right)\right]dF\left(x'\right) \\
EE\text{ Quit} &\quad +\ \xi f(\theta)\sum_{i=1}^{n(x)}\int_{x'\in\mathcal{Q}^{E}(x,i)}\left[\mathbf{J}\left(q_{E}\left(x,i,x'\right)\right)-J\left(x\right)\right]dF\left(x'\right).
\end{aligned}$$

The component $J_F\left(x\right)$ is given by

$$\begin{aligned}
\rho J_{F}\left(x\right) &= \\
UE\text{ Hire} &\quad \phi q(\theta)v\left(x\right)\left[\mathbf{J}\left(h_{U}\left(x\right)\right)-J\left(x\right)\right]\cdot\mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\text{ Threat} &\quad +\ \phi q(\theta)v\left(x\right)\left[\mathbf{J}\left(t_{U}\left(x\right)\right)-J\left(x\right)\right]\cdot\mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\text{ Hire} &\quad +\ \left(1-\phi\right)q(\theta)v\left(x\right)\int_{x\in\mathcal{Q}^{E}(x',i')}\left[\mathbf{J}\left(h_{E}\left(x',i',x\right)\right)-J\left(x\right)\right]dG\left(x',i'\right) \\
EE\text{ Threat} &\quad +\ \left(1-\phi\right)q(\theta)v\left(x\right)\int_{x\notin\mathcal{Q}^{E}(x',i')}\left[\mathbf{J}\left(t_{E}\left(x',i',x\right)\right)-J\left(x\right)\right]dG\left(x',i'\right) \\
\text{Shock} &\quad +\Gamma_{z}\left[\mathbf{J},J\right]\left(x\right).
\end{aligned}$$

Note that, in continuous time, at most one contact is made per instant. That is, either one worker is exogenously separated, or one worker is contacted by another firm, or one worker is met by posting vacancies (at rate $q(\theta)v(x)$), or a shock hits the firm. Note also that we have $\mathbf{J}$ bold everywhere since after any of these events, the firm may want to layoff some workers or exit, and workers may want to quit.

Entry. The expected value of an entrant firm is

$$J_{0}=-c_{e}+\int\mathbf{J}\left(x_{0}\right)d\Pi_{0}\left(z_{0}\right) \tag{15}$$

where x_0 is the state of the entrant firm which includes the random productivity value z_0 drawn from Π_0 and the initial number of workers n_0. The argument of the integral is $\mathbf{J}$ which incorporates the firm's exit decision after observing z_0. With free-entry, in equilibrium $J_0 = 0$.

4.3 Joint value function: Ω

Let $\Omega(x) = J(x) + \sum_{i=1}^{n(x)} V(x,i)$, be the **joint value** of the firm and its employed workers. We also define the joint value before exit/quit/layoff decisions: $\mathbf{\Omega}(x) = \mathbf{J}(x) + \sum_{i=1}^{n(x)} \mathbf{V}(x,i)$. Appendix A shows that simply adding up firm and worker values and applying these definitions delivers the following Bellman equation for the joint value:

$$
\begin{aligned}
\rho\Omega(x) &= y(x) - c(v(x), x) && (16)\\
\text{Destruction} \quad & + \sum_{i=1}^{n(x)} \delta \left[\mathbf{\Omega}(d(x,i)) + U - \Omega(x)\right] \\
\text{Retention} \quad & + \xi f(\theta) \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\mathbf{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega(x)\right] dF\left(x'\right) \\
EE \text{ Quit} \quad & + \xi f(\theta) \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\mathbf{\Omega}\left(q_E\left(x,i,x'\right)\right) + \mathbf{V}\left(h_E\left(x,i,x'\right),i\right) - \Omega(x)\right] dF\left(x'\right) \\
UE \text{ Hire} \quad & + \phi q(\theta) v(x) \left[\mathbf{\Omega}(h_U(x)) - U - \Omega(x)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE \text{ Threat} \quad & + \phi q(\theta) v(x) \left[\mathbf{\Omega}(t_U(x)) - \Omega(x)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE \text{ Hire} \quad & + (1-\phi) q(\theta) v(x) \int_{x \in \mathcal{Q}^E(x',i')} \left[\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \mathbf{V}\left(h_E\left(x',i',x\right),i'\right) - \Omega(x)\right] dG\left(x',i'\right) \\
EE \text{ Threat} \quad & + (1-\phi) q(\theta) v(x) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\mathbf{\Omega}\left(t_E\left(x',i',x\right)\right) - \Omega(x)\right] dG\left(x',i'\right) \\
\text{Shock} \quad & + \Gamma_z\left[\mathbf{\Omega}, \Omega\right](x).
\end{aligned}
$$

Note that this joint value is only written in terms of joint value and worker values. However, it involves both firm and worker decisions.

4.4 Reducing the state space

To make progress on (16), we begin by stating six intermediate results **(R1)-(R6)** which we prove from the four assumptions listed in Section 2.2. These six results establish worker values $\mathbf{V}$ in (16) evolve in the six cases of hiring, retention, layoff, quits, exit and vacancy creation. Since each result extends the logic of the static example we relegate the proofs that claim holds to Appendix A. We then apply these results to (16) such that the only payoff relevant state variables in the vector x are productivity z and size n.

To highlight the structure of the argument, we note a key implication of combining two assumptions: **(A-TR)** (value is transferable between firms and workers) and **(A-MC)** (all changes in contracts must make parties weakly better off). This implies that during *negotiation between the firm and its incumbents* any value lost to one party must accrue to the other, which we already saw in the static example. In other words, the joint value of the firm plus its incumbent workers is invariant across the negotiation. We use this property extensively in the proof. This generalizes pairwise efficient bargaining—commonly used in one-worker-one firm models with linear production—to an environment with multi-worker firms and decreasing returns in production.

It is useful to specify once again the sequence of negotiation for the different types of meetings, as implied by **(A-BP)** and **(A-MC)**.

1. **Meeting with an unemployed worker** - When the firm meets an unemployed worker, *internal negotiation* takes place between the firm and all incumbent workers, in random order. Then *external negotiation* takes place between the firm and the unemployed worker. Because of the take-leave nature of firm offers, no other round of negotiations is needed (we show this formally in Appendix A).

2. **Meeting with an employed worker** - When the firm meets an employed worker, the firm—which is the poaching firm in this case—first *internally negotiates* with all incumbent workers. It then *externally negotiates* with the matched worker. Following this, the current firm to which the worker belongs *internally negotiates* with all of its workers, including the matched worker. If the poaching (current) firm has made a higher offer to the worker in external (internal) negotiations, then the matched worker goes to the poaching (current) firm. Finally, both the poaching and current firm may *internally renegotiate* with their (possibly new) set of incumbent workers.

We now state the six conditions that we apply to (16). In Appendix A, we prove how each of them is implied by the assumptions of Section (2.2) .

(C-RT) **Retentions and Threats.** First, if firm x meets an unemployed worker and the worker is not hired but only used as a threat, then the joint value of coalition x does not change since threats only redistribute value within the coalition. Second, and similarly, when firm x uses worker i' from firm x' as a threat, the joint value of coalition x does not change. Third, when firm x meets worker i' at x' and the worker is retained by firm x', the joint value of coalition x' does not change. Respectively, these imply

$$\mathbf{\Omega}\left(t_U\left(x\right)\right)=\Omega(x)\quad,\quad\mathbf{\Omega}\left(t_E\left(x',i',x\right)\right)=\Omega(x)\quad,\quad\mathbf{\Omega}\left(r\left(x',i',x\right)\right)=\Omega(x')$$

(C-UE) *UE* **Hires.** An unemployed worker that meets firm x is hired when $x\in\mathcal{A}$. This consists of firms that have a joint value after hiring that is higher than the pre-hire joint value plus the outside value of the hired worker: $\mathcal{A}=\{x|\mathbf{\Omega}(h_U(x))-\Omega(x)\geq U\}$. The new hire receives her outside value: the value of unemployment

$$\mathbf{V}\left(h_U\left(x\right),i\right)=U.$$

(C-EE) *EE* **Hires.** An employed worker i' at firm x' that meets firm x is hired when $x\in\mathcal{Q}^E\left(x',i'\right)$. This consists of firms that have a higher marginal joint value than that of the current firm:

$$\mathcal{Q}^E\left(x',i'\right)=\left\{x\middle|\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega\left(x\right)\geq\Omega(x')-\mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)\right\}.$$

The new hire receives her outside value: the marginal combined value at the current firm

$$\mathbf{V}\left(h_E\left(x',i',x\right)\right)=\Omega\left(x'\right)-\mathbf{\Omega}\left(q_E\left(x',i',x\right)\right).$$

(C-EU) *EU* **Quits and Layoffs.** An employed worker i at firm x quits to unemployment when $(x,i)\in\mathcal{Q}^U$. This consist of states x such that the marginal joint value is below that value of unemployment:

$$\mathcal{Q}^U=\left\{(x,i)\middle|\Omega\left(\hat{s}_{q1}\left(x,i\right)\right)+U>\Omega\left(\hat{s}_{q0}\left(x,i\right)\right)\right\}$$

where

$$\begin{aligned}\hat{s}_{q1}\left(x,i\right)&=s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=1\right]\right)\right),\\ \hat{s}_{q0}\left(x,i\right)&=s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=0\right]\right)\right).\end{aligned}$$

The first expression being where worker i quits, and the second where worker i does not. Similarly an *EU* layoff will be chosen by the firm when $(x,i)\in\mathcal{L}$, where:

$$\mathcal{L}=\left\{(x,i)\middle|\Omega\left(\hat{s}_{\ell1}\left(x,i\right)\right)+U>\Omega\left(\hat{s}_{\ell0}\left(x,i\right)\right)\right\}$$

where

$$\hat{s}_{\ell 1}(x,i) = s\left(x,(1-[\ell(x);\ell_i(x)=1])\circ(1-q_U(x))\right),$$
$$\hat{s}_{\ell 0}(x,i) = s\left(x,(1-[\ell(x);\ell_i(x)=0])\circ(1-q_U(x))\right).$$

The first expression being where worker i is laid off, and the second where worker i is not.

(C-X) **Exit.** A firm x exits when $x \in \mathcal{E}$. This consists of the states in which the total outside value of the firm and its workers is larger than the joint value of operation:

$$\mathcal{E} = \left\{x \middle| \vartheta + n\left(s\left(x,\kappa(x)\right)\right)\cdot U > \Omega\left(s\left(x,\kappa(x)\right)\right)\right\}.$$

(C-V) **Vacancies.** The expected return to a matched vacancy $R(x)$ depends only on the joint value, and so the firm's optimal vacancy policy $v(x)$ depends only on the joint value. The policy $v(x)$ solves

$$\max_v \; q(\theta)vR(x) - c(v,x),$$

where the expected return to a matched vacancy is

$$\begin{aligned} R(x) &= \phi \underbrace{\left[\boldsymbol{\Omega}\left(h_U(x)\right) - \Omega(x) - U\right]\cdot \mathbb{I}_{\{x\in\mathcal{A}\}}}_{\text{Return from unemployed worker match}} \\ &+ (1-\phi)\underbrace{\int_{x\in\mathcal{Q}^E(x',i')}\left\{\left[\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega(x)\right]-\left[\Omega\left(x'\right)-\boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)\right]\right\}dG\left(x',i'\right)}_{\text{Expected return from employed worker match}}. \end{aligned}$$

(C-E) **Entry.** A firm enters if and only if

$$\int \boldsymbol{\Omega}(x_0)\, d\Pi_0(z) \geq c_0 + n_0 U.$$

Summarizing (C). The substantive result is that all firm and worker decisions and employed workers' values can be expressed in terms of joint value Ω and exogenous worker outside option U. Since $\Omega(x)$ and U are independent of any state variables that determines how surpluses are split, they only depend on (n,z). We can therefore write (16) in terms of only (n,z). All details are in Appendix A).

Applying (C). Substituting conditions **(C)** into (16) we can express the joint values pre- and post-separation and exit decisions as follows. First, the exit and separation decisions are characterized by

$$\begin{aligned} \mathbf{\Omega}(n,z) &= \mathbb{I}_{\{x\in\mathcal{E}\}}\Big\{\vartheta + nU\Big\} + \mathbb{I}_{\{x\in\mathcal{Q}^{U}\}}\Big\{\mathbf{\Omega}(n-1,z)+U\Big\} +, \mathbb{I}_{\{x\notin\mathcal{Q}^{U}\cup\mathcal{E}\}}\Omega(n,z), \qquad (17)\\ \text{where} \quad \mathcal{E} &= \big\{n,z\big|\vartheta + nU > \Omega(n,z)\big\}, \\ \mathcal{Q}^{U} &= \big\{n,z\big|\mathbf{\Omega}\left(n-1,z\right)+U > \Omega\left(n,z\right)\big\}. \end{aligned}$$

The first expression is the value of exit. A firm that does not exit, chooses whether to separate with a worker or not. If separating with a worker, the firm re-enters (17) with $\mathbf{\Omega}(n-1,z)$, having dispatched with a worker with value U, and again choosing whether to exit, fire another worker, or continue. Iterating on this procedure delivers

$$\mathbf{\Omega}(n,z) = \max\left\{\vartheta + nU\,,\ \max_{s\in[0,\ldots,n]}\Omega(n-s)+sU\right\}. \qquad (18)$$

Second, the post-exit/separation decision joint value is given by the Bellman equation

$$\begin{aligned} \rho\Omega\left(n,z\right) &= \max_{v\geq 0} y\left(n,z\right) - c\left(v,n,z\right) \qquad (19)\\ \text{Destruction} \quad &+ \quad \delta n\Big\{\big(\mathbf{\Omega}\left(n-1,z\right)+U\big) - \Omega\left(n,z\right)\Big\} \\ UE\ \text{Hire} \quad &+ \quad \phi q(\theta)v\cdot\mathbb{I}_{\{(n,z)\in\mathcal{A}\}}\cdot\Big\{\mathbf{\Omega}\left(n+1,z\right) - \big(\Omega\left(n,z\right)+U\big)\Big\} \\ EE\ \text{Hire} \quad &+ \quad (1-\phi)\,q(\theta)v\int_{(n,z)\in\mathcal{Q}^{E}(n',z')}\Big\{\left[\mathbf{\Omega}\left(n+1,z\right)-\Omega\left(n,z\right)\right] - \left[\Omega\left(n',z'\right) - \mathbf{\Omega}\left(n'-1,z'\right)\right]\Big\}dG\left(n',z'\right) \\ \text{Shock} \quad &+ \quad \Gamma_z\left[\mathbf{\Omega},\Omega\right]\left(n,z\right), \\ \text{where} \quad \mathcal{A} &= \big\{n,z\big|\mathbf{\Omega}\left(n+1,z\right)\geq\Omega\left(n,z\right)+U\big\}, \\ \mathcal{Q}^{E}\left(n',z'\right) &= \big\{n,z\big|\mathbf{\Omega}\left(n+1,z\right)-\Omega\left(n,z\right)\geq\Omega\left(n',z'\right)-\mathbf{\Omega}\left(n'-1,z'\right)\big\}. \end{aligned}$$

Finally, firms enter if and only if

$$\int \mathbf{\Omega}\left(n_e,z\right)d\Pi_0(z) \geq c_e + n_e U. \qquad (20)$$

This condition pins down the entry rate per unit of time.[17]

We now take the continuous work-force limit of equations (18) and (19) and specify a process for Γ_z.

[17] Recall that $J_0 = -c_e + \int \mathbf{J}(x_0)d\Pi(z_0)$. Given $\Omega(n_0,z_0) = J(n_0,z_0) + n_0U$, we have $J_0 = -c_e - n_0U + \int \mathbf{\Omega}(n_0,z_0)d\Pi(z_0)$. Free-entry implied $J_0 = 0$, which delivers (20).

4.5 Continuous workforce limit

Up until now the economy has featured a continuum of firms, but an integer-valued workforce. We now take the continuous workforce limit by defining the 'size' of a worker—which is 1 in the integer case—and taking the limit as this approaches zero. Specifically, denote the "size" of a worker by Δ, such that $n = \Delta m$ where m is the old integer number of workers. Now define $\Omega^{\Delta}(n,z) = \Omega(n/\Delta, z)$, and likewise define $y^{\Delta}(n,z)$ and $c^{\Delta}(v,n,z) = c(v/\Delta, n/\Delta, z)$. We also define $b^{\Delta} = b/\Delta$ and $\vartheta^{\Delta} = \vartheta/\Delta$. These imply, for example, that $\Omega(m,z) = \Omega^{\Delta}(m\Delta, z)$. Substituting these terms into (18) and (19), and taking the limit $\Delta \to 0$, while holding $n = m\Delta$ fixed, we would obtain a version of (21) in which all functions have the Δ super-script notation, here removed for clarity (see Appendix A). We also specialize the productivity to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$.

The result is a HJB for the joint value *conditional on (weakly) hiring*:

$$\begin{aligned} \rho\Omega(n,z) = \max_{v\geq 0} \quad & y(n,z) - c(v,n,z) \\ \text{Destruction} \quad & -\delta n[\Omega_n(n,z) - U] \\ UE \text{ Hire} \quad & +\phi q(\theta) v \left[\Omega_n(n,z) - U\right] \\ EE \text{ Hire} \quad & +(1-\phi) q(\theta) v \int \max\left\{\Omega_n(n,z) - \Omega_n(n',z')\,,\, 0\right\} d\tilde{G}(n',z') \\ \text{Shock} \quad & +\mu(z)\Omega_z(m,z) + \frac{\sigma(z)^2}{2}\Omega_{zz}(m,z). \end{aligned} \tag{21}$$

The boundary conditions for the value of hiring are therefore that the value is interior to the exit boundary and the separation boundary:

$$\begin{aligned} \text{Exit boundary} \quad & \Omega(n,z) \geq \vartheta + nU \\ \text{Separation boundary} \quad & \Omega_n(n,z) \geq U \end{aligned}$$

When the value hits the separation boundary, separations occur such that the value remains on the boundary. When the value hits the exit boundary, the firm dissolves and the value equals $\Omega(n,z) = \vartheta + nU$.

Note that we no longer have $\boldsymbol{\Omega}$ terms. Since the value that we keep track of is that of a hiring firm subject to boundary conditions, then $\boldsymbol{\Omega} = \Omega$. This also allows for the simplification of the 'Shock' terms, as we noted when discussing (14).

4.6 Surplus formulation

Finally, as we did in the static example of Section 3, we can reformulate (21) in terms of total surplus:

$$S(z,n) := \Omega(z,n) - nU \quad , \quad S_n(z,n) = \Omega_n(z,n) - U, \quad , \quad S_z(z,n) = \Omega_z(z,n) \quad , \quad S_{zz}(z,n) = \Omega_{zz}(z,n).$$

In doing so note that marginal surplus $S_n(z',n')$ at a competitor is sufficient to compute the return to a vacancy meeting that firm. We therefore directly compute the value of a vacancy using $G(S_n')$: the employment weighted distribution of marginal surplus $S_n(n',z')$ in the economy. With these definitions we arrive at

$$\begin{aligned}
\rho S\left(z,n\right) &= \max_{v\geq 0} \quad y\left(z,n\right) - nb - \delta n S_n\left(z,n\right) \\
&+ \underbrace{\left[\phi S_n\left(z,n\right) + \left(1-\phi\right)\int_0^{S_n(z,n)} S_n\left(z,n\right) - S_n'\, dG\left(S_n'\right)\right]}_{\text{Return on a vacancy } R(z,n)=\tilde{R}(S_n(z,n))} q(\theta)v - c(v,n) \\
&+ \mu(z) S_z\left(z,n\right) + \frac{\sigma^2(z)}{2} S_{zz}\left(z,n\right)
\end{aligned} \tag{22}$$

subject to the two boundary conditions for exit ($S\left(z,n\right) \geq c_f$) and separation ($S_n\left(z,n\right) \geq 0$).

Discussion The formulation of the HJB equation for the total surplus is strikingly straight-forward. We observe the productive surplus of output minus the utility that would accrue to the firm's workers in unemployment. At rate δ one of the firm's n workers is exogenously removed from the firm. At rate $\phi q(\theta)v$ the firm meets an unemployed worker, who is hired with a unit effect on the firm's marginal surplus. At rate $(1-\phi)q(\theta)v$ the firm meets an employed worker who is hired if the marginal surplus of the worker's existing match—which his current firm-worker group would be will to transfer to him to get him to stay—is less than the marginal surplus of the firm. The firm again makes a gain according to their marginal surplus, but loses the payment to the worker which is equal to the competitor's marginal surplus. Finally, the expected innovation to productivity leads to changes in total surplus.

The Bellman equation is a natural extension of the kinds of expressions of firm value one finds in single worker job ladder models (Postel-Vinay and Robin, 2002; Lise and Robin, 2017). For example, the following three equations define values in a single-worker firm version of our model with no productivity shocks and $\vartheta = 0$:

$$\begin{aligned}
\text{Free entry} \quad c_e &= \int J(z) d\Pi_0(z) \\
\text{Value} \quad \rho J(z) &= \max_v \left[\phi S(z) + (1-\phi) \int_0^{S(z)} S(z) - S' \, dG(S') \right] q(\theta) v - c(v) \\
\text{Surplus} \quad S(z) &= \frac{y(z,1) - b}{\rho + \delta}
\end{aligned}$$

However since surplus depends only on exogenous productivity z, the expected return to a vacancy is effectively computed by integrating over $F(z)$:

$$\text{Value} \quad J(z) = \max_v \left[\phi S(z) + (1-\phi) \int_0^{S(z)} S(z) - S(z')' \, dF(z') \right] q(\theta) v - c(v)$$

Here, the integral term—crucial to determining the firm's incentives to post vacancies with on-the-job search—depends on the *endogenous* distribution of marginal values. G is an equilibrium outcome.

Properties of the joint surplus. We can establish some basic properties of S under standard assumptions on the production function, vacancy cost, and productivity process. Suppose (i) productivity follows a geometric Brownian motion $\mu(z) = \mu \cdot z$, $\sigma(z) = \sigma \cdot z$, , (ii) the vacancy cost function is isoelastic in vacancies only $c(v) = c_0 v^{1+\gamma}$, and (iii) the production function satisfies $y_z > 0, y_n < 0, y_{nn} < 0, y_{zn} > 0$ all of which are satisfied by $y(z,n) = zn^\alpha$ with $\alpha \in (0,1)$.

Then we obtain the following properties of $\Omega(z,n)$:

1. Increasing and concave in employment: $\Omega_n > 0$, $\Omega_{nn} < 0$
2. Increasing in productivity: $\Omega_z > 0$
3. Supermodular in productivity and labor $\Omega_{zn} > 0$
4. Net employment growth is increasing with productivity z and decreasing with size n.

These results are proven in Appendix B.

4.7 Steady-state equilibrium

A stationary, free-entry equilibrium consists of a total surplus function $S(z,n)$, a positive mass of entrants $M_e > 0$, a positive measure of firms $H(z,n)$, a vacancy policy $v(z,n)$, and a law of motion of firm level employment $\dot{n}(n,z)$ that satisfy the following conditions.

Total surplus $S(z,n)$ satisfies (22), where the distribution of marginal surplus $G(S)$ is consistent with $H(z,n)$:

$$G(S) = \int \mathbb{1}\left[S_n(n,z) = S\right] dH(z,n).$$

The law of motion for firm employment is

$$\dot{n}(n,z) = \begin{cases} -n & n < n_E^*(z) \\ v(z,n)\left[\phi + (1-\phi)G(S_n(z,n))\right] - \delta n & n \in \left[n_E^*(z), n_S^*(z)\right) \\ n^*(z) - n & n \geq n^*(z). \end{cases}$$

The vacancy policy satisfies the first order condition:

$$c_v(v(z,n),n) = q(\theta)\left[\phi S_n(z,n) + (1-\phi)\int_0^{S_n(z,n)} S_n(z,n) - S' dG(S')\right].$$

and free-entry implies that the entry condition holds with equality

$$c_e = \int S(z,n_e) d\Pi_e(z).$$

The measure of firms over $H(z,n)$ satisfies

$$0 = \dot{n}\left(n,z\right) H_n\left(n,z\right) - \mu\left(z\right) H_z\left(n,z\right) - \frac{\sigma\left(z\right)^2}{2} H_{zz}\left(n,z\right)$$

for general (n,z). At n_e, the number of workers at an entering firm,

$$0 = \dot{n}\left(n_e,z\right) H_n\left(n_e,z\right) - \mu\left(z\right) H_z\left(n,z\right) - \frac{\sigma\left(z\right)^2}{2} H_{zz}\left(n,z\right) + M_e \Pi_e(z).$$

Aggregate employment is then $\mathtt{n} = \int nH(z,n)$ so aggregate unemployment is $\mathtt{u} = \bar{\mathtt{n}} - \mathtt{n}$. An equilibrium therefore requires that $\theta = \mathtt{v}/\mathtt{s}$ under aggregate vacancies $\mathtt{v} = \int v(z,n) dH(z,n)$, aggregate search effort $\mathtt{s} = \mathtt{u} + \xi \mathtt{n}$, and the conditional meeting probability $\phi = \mathtt{u}/\mathtt{s}$.

Note that we do not need to directly write down a law of motion for unemployment, since this is implied by integrating employment among firms. This is like writing down the from the point of view of the Bureau of Labor Statistics' *Current Employment Statistics* rather than the *Current Population Survey*.

Equilibrium is determined as follows. Consider holding the distribution of marginal surpluses $G(S)$ fixed. The determination of equilibrium M_e and θ follows from Hopenhayn (1992) with an additional step. We can normalize the mass of vacancies in the economy to one. Then $\theta(\mathtt{u}) = 1/[\mathtt{u} + \xi[\bar{\mathtt{n}} - \mathtt{u}]]$, and $\phi(\mathtt{u}) = \mathtt{u}/[\mathtt{u} + \xi[\bar{\mathtt{n}} - \mathtt{u}]]$. Note that $\theta'(\mathtt{u}) < 0$ such that $q'(\mathtt{u}) > 0$ and $\phi(\mathtt{u}) > 0$. When unemployment is higher, the overall matching rate is higher (q), and the conditional probability of meeting an unemployed worker is higher (ϕ). Now note that $S(z, n)$ is increasing in both ϕ (cheaper to hire from unemployment) and q (cheaper to hire in general). We can therefore determine $\mathtt{u}^*$ such that the free-entry condition holds without computing $H(z, n)$. Given policies we can then simply find M_e such that the mass of vacancies in the economy is indeed one. The added step is then checking that the implied distribution $G(S)$ is correct, and iterating on this procedure until it is.

In practice, as in Hopenhayn and Rogerson (1993), this is simplified by noting that the magnitudes of c_e are somewhat arbitrary. We therefore can skip an intermediate step, fixing $\mathtt{u}$ such that the unemployment rate $\mathtt{u}/(\bar{\mathtt{n}} + \mathtt{u})$ is equal to the US unemployment rate, and backing out the implied c_e such that free-entry holds.

Additional details regarding the computation of the equilibrium of the model are contained in Appendix D.

5 Comparative statics

5.1 Vacancy policy and hiring costs

From (22), the first order condition for the firm's vacancy decision gives

$$\begin{aligned} qR(S_n(z,n)) &= c_v(v,z,n), \\ \text{where} \quad R(S) &= \phi S + (1-\phi) \int_0^S S - S' \, dG\left(S'\right) \end{aligned}$$

The return on a vacancy is constant in v, and is a function only of the marginal surplus at the firm. Note that $R(S)$ is increasing and convex in the marginal surplus:

$$\begin{aligned} R'(S) &= \phi \cdot 1 + (1-\phi)\Big[\underbrace{G(S)}_{\text{Intensive}} + \underbrace{(S-S)g(S)}_{\text{Extensive}} \Big] > 0 \\ R''(S) &= (1-\phi)g(S) > 0 \\ \lim_{S\to\infty} R'(S) &= 1. \end{aligned}$$

An increase in S increases the return from hiring an unemployed worker one-for-one. It also increases the return on hiring any employed worker one-for-one, for $G(S)$ mass of potential hires. Note that this intensive margin return is the only way in which an increase in S increases the return on a vacancy through poaching. On the extensive margin, increasing S increases the set of firms from which the firm may poach, but since the marginal surplus of the traded worker is approximately the same at both firms, there is no additional value.

Now consider an increase in S and a special case of the cost function with $c(v,n,z) = c(v,S)$. Then by the Implicit Function Theorem

$$qR(S) = c_v(v,S) \quad \Longrightarrow \quad v'(S) = \frac{qR'(S) - c_{vS}(v,S)}{c_{vv}(v,S)} = \frac{q\left[\phi + (1-\phi)G(S)\right] - c_{vS}(v,S)}{c_{vv}(v,S)}.$$

A sufficient condition for $v'(S) > 0$ is $c_{vS} < 0$. If firms that have a higher marginal productivity are also more efficient in vacancy posting then this is true.

Alternatively we could think of the implied costs of growing the firm. Consider the probability of hiring one worker: $p = q(\theta)v[\phi + (1-\phi)G(S)]$. We can approximate the firm's expected growth rate $\tilde{g} = \frac{p(n+1)+(1-p)n-n}{n}$ such that $p = \tilde{g}n$. We can write down the vacancies required and cost of attaining $\tilde{g}$ for a firm with (S,n):

$$\begin{aligned} v(\tilde{g},S,n) &= \frac{\tilde{g}n}{q(\theta)\left[\phi + (1-\phi)G(S)\right]}, \\ c(\tilde{g},S,n) &= c\left(\frac{\tilde{g}n}{q(\theta)\left[\phi + (1-\phi)G(S)\right]}, n\right). \end{aligned}$$

The effective (positive) adjustment cost depends positively on the growth rate, and ambiguously on n depending on the size of $c_n < 0$. The effective (positive) adjustment cost is also decreasing in the firm's marginal surplus and determined by two equilibrium objects: overall market tightness, $q(\theta)$ and, at the margin, the gradient of the economy's distribution of marginal surplus $g(S)$.

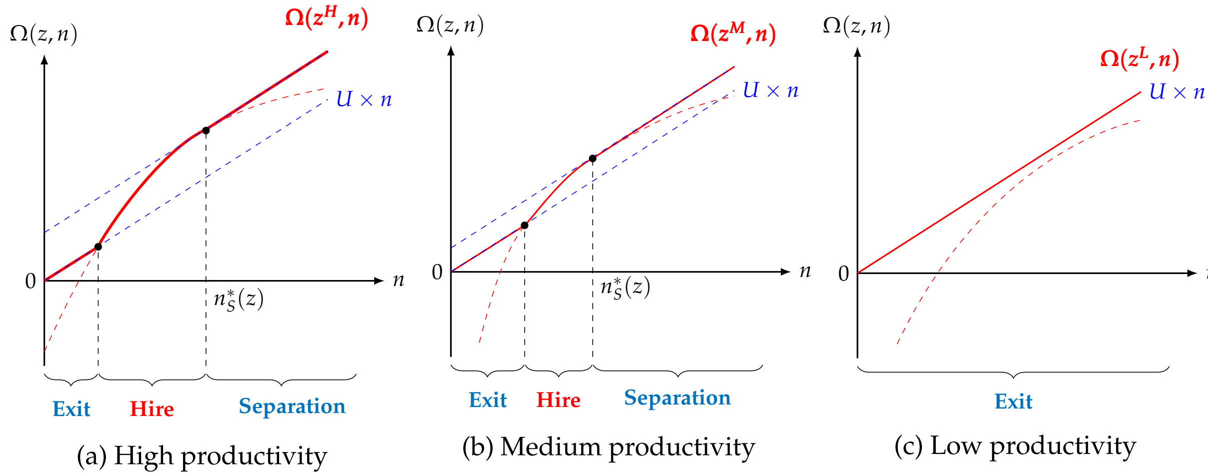


Figure 3: Values of exit, hiring and separation for fixed levels of productivity z

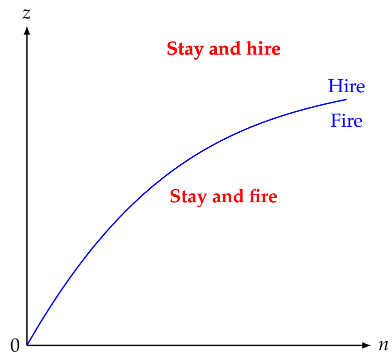
Notes: Panel (a) gives the surplus value $\Omega(z,n)$ of a high productivity z^H firm. The red dashed line gives the value of hiring minus the fixed cost of operation $\Omega(z,n)-\vartheta$. The lower blue dashed line extending from the origin gives the joint value if the firm were to exit: $U \times n$. The upper blue dashed line has the same slope U, and lies tangent to Ω such that at the cut-off n^*, $\Omega_n(z,n^*)=U$. The solid red line is the value $\boldsymbol{\Omega}(z,n)=\mathbb{1}_{\{n<n^*\}}\max\{nU,\Omega(z,n)-\vartheta\}+\mathbb{1}_{\{n\geq n^*\}}\left[\Omega(n^*,z)+(n-n^*)U\right]$. Panel (b) replicates the first, but for a lower $z^M<z^H$. Under a lower productivity, the exit and separation regions increase, while the hiring region shrinks. Panel (c) replicates the first, but for an even lower $z^L<z^M$ under which it is optimal for the firm to exit for all n.

This is new and makes clear the role of on-the-job search as an endogenous adjustment cost. Compare this to the effective cost function in Gavazza, Mongey, and Violante (2018) which is convex in the firm's growth rate but depends on the distribution of firms in the economy only through the 'price' $q(\theta)$, or to the standard convex adjustment cost in firm dynamics models which depend only on the hiring rate.

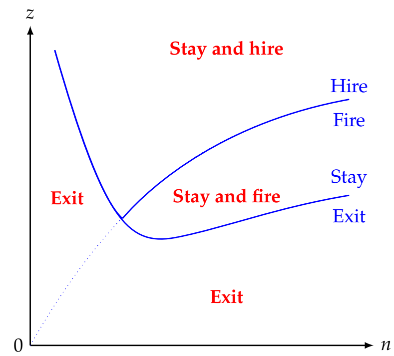
5.2 Values of exit, hiring and separation

Figure 3 provides examples of the firm's policies for alternative levels of productivity. The red dashed line gives the value of hiring minus the fixed cost of operation $\Omega(z,n)-\vartheta$. The lower blue dashed line extending from the origin gives the total value of unemployment to the firms' employees: $U \times n$. The exit threshold $n_E^*(z)$ is determined by $\Omega(z,n_E^*(z))-\vartheta = Un$. As $\Omega(z,n)$ falls along with z, the cut-off $n_E^*(z)$ increases. The separation threshold $n_S^*(z)$ is determined by the condition that the *marginal* value is equal to the value of unemployment U, that is $n_S^*(z)$ is determined by the point at which a line parallel to $U \times n$ is tangent to $\Omega(n,z)$. If $n > n_S^*(z)$, the firm immediately fires $n-n_S^*(z)$ workers who receive value U such that the total firm value, given by the solid red line, is as in equation (18):

$$\boldsymbol{\Omega}(z,n)=\Omega(z,n_S^*(z))+(n-n_S^*(z))\,U \tag{23}$$



(a) No fixed cost $c_f = 0$



(b) Fixed cost $c_f > 0$

Figure 4: Regions of exit, hiring and separation by (n, z)

Notes: Panel (a) gives the following three regions for a case with no fixed cost of operation ($c_f = 0$): Stay and hire, Stay and fire, Exit. The upper solid line gives the boundary between hiring and firing. This is equivalent to $n^*(z)$ from Figure 3. The lower dashed line gives the boundary between staying and exiting. Panel (c) shows the effect of a positive cost of operation ($c_f > 0$) on the Stay-Exit boundary.

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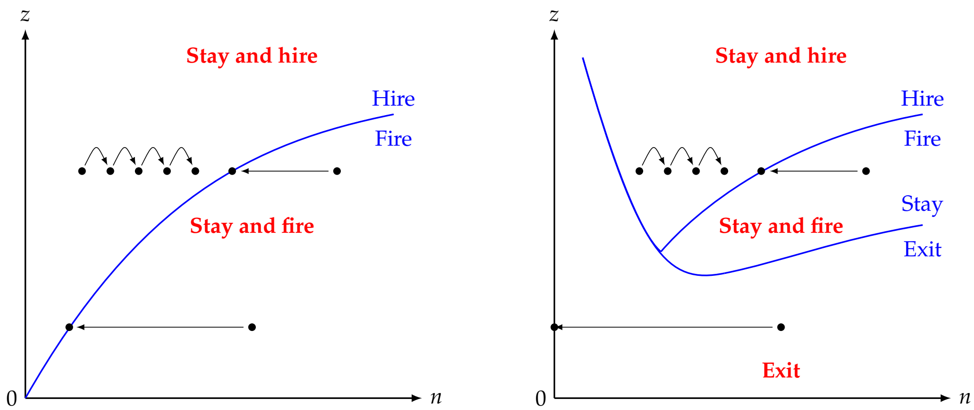
5.3 Regions for exit, hiring and separation

Figure 4 takes plots out the functions that determine the exit frontier $n_E^*(z)$ and separation frontier $n_S^*(z)$ for all values of z. Panel (a) considers the model without fixed costs. Introducing fixed costs leads small firms to exit, but small and highly productive firms stay and hire. There exist cases of z such that if n is very low, the present discounted cost of growing exceeds the value of exit, so the firm exits. If n were larger, then the firm instead would decide to stay and grow. If n were larger still, the firm fires workers until $\Omega(z, n) = U$. Finally, if n were even larger then as $n \times U$ increases linearly but $\Omega(z, n)$ increases at a decreasing rate, there reaches a point at which the firm exits. Note, however, that it is impossible for the firm—under a continuous process for productivity shocks—to ever enter this region after being born small. Figure 5 captures these dynamics.



(a) No fixed cost $c_f = 0$ (b) Fixed cost $c_f > 0$

Figure 5: Firm dynamics

Notes: This figure augments Figure 4 by including examples of hypothetical firm paths, in each case keeping productivity fixed. A firm (black dot) that begins in the separation region jumps to the separation frontier, firing $n - n^*(z)$ workers. A subsequent decline in productivity smoothly moves the firm around the separation frontier. A firm that begins in the hiring region smoothly increases their employment toward $n^*(z)$.

6 Calibration (Preliminary)

The assumed frequency of the model is monthly, which we aggregate to annual frequency measures for firm dynamics.

6.1 Functional forms

We assume that vacancy costs $c(v, n) = c\left(\frac{v}{n}\right)^{\gamma+1} v$ as in Kaas and Kircher (2015). The average cost of a vacancy is therefore increasing in the firm's vacancy rate. The interpretation being that it is more costly for a small firm to hold open many vacant positions. We assume that the production function is $y(z, n) = zn^{\alpha}$.[18] We assume that the aggregate matching function is Cobb-Douglas with elasticity with respect to vacancies β such that $f(\theta) = \chi\theta^{\beta}$ and $q(\theta) = \chi\theta^{1-\beta}$. We assume that the entry distribution is Pareto in levels with a lower bound of zero and shape parameter ζ.

Finally, we add some exogenous exit to the model. Firms exit the economy exogenously at the death rate d. We plan to decompose firm exits into endogenous and exogenous exit.

[18] Note that throughout we refer to this as a *production* function, but do not distinguish between revenue and output. This could instead represent a decreasing returns to scale *revenue* function of a firm with market power that operates a constant returns to scale technology but internalizes the downward sloping demand curve for its goods.

Table 4: Pre-set parameters

	Parameter	Value	Target
ρ	Discount rate	0.004	5% annual real interest rate
χ	Matching efficiency	0.1	Normalization
β	Elasticity of matches w.r.t. vacancies	0.5	Petrongolo and Pissarides (2001)
c_f	Fixed cost of production	1	Normalization

6.2 Externally calibrated

We first preset a few parameters to standard values in the literature. Table 4 summarizes these parameter values. We set the discount rate to match an annual real interest rate of five percent. We normalize, without loss of generality, the fixed cost of production. Absent data on vacancies, matching efficiency is isomorphic to the scalar cost of hiring, c, so we normalize $\chi = 1$. We set $\theta = 0.5$ based on standard values in the literature.

6.3 Internally calibrated

This leaves us with 10 parameters to determine:

$$\psi = \{\mu, \sigma, \zeta, d, \alpha, c, \gamma, b, \zeta, \delta\}$$

We estimate these by GMM targeting moments of the idiosyncratic productivity process of firms, firm reallocation measures and worker reallocation measures. While the parameters in ψ are jointly determined, some moments are particularly informative about some parameters. In particular, the unweighted exit rate is informative about the drift of productivity, μ. A more negative drift means that firms fall below the exit threshold at faster rate. The annual standard deviation of TFP shocks is informative about the standard deviation of productivity shocks, σ. The shape of the entry distribution, ζ, maps into the cross-sectional standard deviation of TFP. If entrants enter more dispersed this is reflected in a more dispersed stationary distribution. While μ governs the employment-unweighted exit rate, firms that fall below the endogenous exit frontier are on average small. The exogenous exit rate d is set such that some large firms also exit, i.e. to match the employment-weighted exit rate.

Table 5 summarizes the estimated parameter values and the model fit with respect to the targeted moments. Moments regarding productivity dynamics are taken from Decker, Haltiwanger, Jarmin, and Miranda (2018). Moments regarding firm dynamics are averages

Table 5: Estimated parameters

	Parameter	Value	Target	Data	Model
μ	Drift of productivity	-0.007	Annual exit rate, unweighted	0.076	0.099
σ	Std. dev. of productivity shocks	0.043	Std. dev. of cross-sectional TFP	0.27	0.500
ζ	Shape of entry distribution	2.753	Std. dev. of cross-sectional TFP of entrants	0.37	0.357
d	Exogenous death rate of firms	0.002	Annual exit rate, weighted	0.029	0.032
α	Curvature of production	0.644	Employment share $n \geq 500$	0.518	0.576
c	Scalar in hiring cost	5066.9	Monthly UE hazard	0.218	0.129
γ	Curvature of hiring cost	9.807	Annual job creation	0.136	0.151
b	Flow value of leisure	0.388	Monthly EU hazard	0.012	0.010
ϕ	Relative search efficiency	0.180	Monthly EE hazard	0.018	0.002
δ	Exogenous separation rate	0.005	Monthly EU hazard	0.012	0.010

Notes: Moments regarding productivity dynamics are taken from Decker, Haltiwanger, Jarmin, and Miranda (2018). Moments regarding firm dynamics are averages of HP-filtered Census BDS data between 2013–2016,[20] Moments regarding worker dynamics are averages of HP-filtered CPS data between 2013–2016, where matched samples are used and workers are over 16 years of age.

of HP-filtered Census BDS data between 2013–2016,[19] Moments regarding worker dynamics are averages of HP-filtered CPS data between 2013–2016, where matched samples are used and workers are over 16 years of age.

The model fits the data well with the signficant exception that implied job finding rates are too high relative to the data. Both the monthly UE hazard, and EE hazard exceed those found in the data.

6.4 Non-targetted moments - Net poaching in the cross-section

In Table 6 we take the quarterly net poaching rates from Haltiwanger, Hyatt, Kahn, and McEntarfer (2018) and Haltiwanger, Hyatt, and McEntarfer (2018) featured in Table 2 and compares them with those in the model. First, as in the data, gross EE flows are significantly larger than net flows, although this ratio is even larger in the data. Second, as in the data, there is a *negative size ladder*—net poaching is positive at small firms and negative at large firms—and a *positive productivity ladder*—net poaching is negative at low productivity firms and positive at high productivity firms.

Caveat Given the issues with the current calibration of the model, we take the sign rather than the magnitude of these moments to be indicative of the potential of the model to rationalize the data.

6.5 Non-targetted moments - Worker flows following a productivity shock

To be completed

[19]When relevant, the moments on firm dynamics refer to firms and not establishments.

	A. Size		B. Productivity	
	Small $n < 50$	Large $n \geq 500$	Low $Q1$	High $Q5$
Data				
Gross *EE* hires	8.9	7.7	7.2	6.5
Gross *EE* separations	8.7	7.8	8.0	5.7
(i) Net poaching	0.2	-0.1	-0.8	0.8
(ii) Gross-Net ratio	44	78	10	8
Model				
Gross *EE* hires	4.1	3.5	3.2	3.9
Gross *EE* separations	3.4	3.7	4.0	3.5
(i) Net poaching	0.7	-0.2	-0.8	0.4
(ii) Gross-Net ratio	5	17	5	9

Table 6: Non-targetted moments - Employment-Employment Worker Flows

Notes: All data are quarterly. Data taken from Table 1, see notes to Table 1 for details. All data are expressed as rates relative to total employment in the given cell. *Net poaching* is equal to *Gross EE hires* minus *Gross EE separations*. *Gross-Net ratio* is the mean of *Gross EE hires* and *Gross EE separations* divided by the absolute value of *Net poaching*.

7 Counterfactual economies

To be completed

8 Conclusion

In this paper we have set out a new framework for worker reallocation and firm dynamics. Consistent with the data—and novel with respect to previous multi-worker firm models of firm dynamics—firms hire from both employment and unemployment. The model features a *marginal surplus ladder*. Firms with a higher marginal productivity, and so higher marginal surplus, find it easier to hire, post more vacancies and grow more quickly. The effective cost of hiring is therefore endogenously cheaper for more productive firms, while unproductive firms spend considerable resources simply keeping their employment level constant as workers are poached away, and so lower their optimal size.

We will study the extent to which these dynamics are important for the quantitative role of search frictions in generating worker misallocation. On-the-job search reallocates workers to more productive firms and makes it more expensive to be maintain a workforce when unproductive. Neither force is present in a firm-dynamics model that accounts only for net flows and the empirical growth rate distribution via convex (net) employment adjustment costs.

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APPENDIX

This Appendix is organized as follows. Section A provides extensive details on the derivation of the joint surplus $\Omega(n, z)$. Section B provides further mathematical details and derivations for some of the results in the paper. Section C provides additional figures and tables. Section D details the algorithms used in the paper.

A Derivation of joint value

To arrive at $\Omega(z, n)$ (equation X) we proceed in a number of steps in the following subsections in which we:

(A.1) Combine values of workers and firm

(A.2) Show that the **Assumptions A** in the main text imply **Conditions C1-C6**

(A.3) Apply **Conditions C1-C6** to the joint worker-firm value

(A.4) Show that we can reduce the state-space to (n, z)

(A.5) Derive the continuous workforce limit

A.1 Combining worker and firm values

We start by computing the summ of the workers' values at a particular firm. Summing values of all the employed workers

$$\begin{aligned}
\rho \sum_{i=1}^{n(x)} V(x,i) &= \sum_{i=1}^{n(x)} w(x,i) \\
\text{Destructions} &\quad + \sum_{i=1}^{n(x)} \delta \left[U - V(x,i) \right] \\
\text{Retentions} &\quad + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r\left(x,i,x'\right),i\right) - V(x,i) \right] dF\left(x'\right) \\
EE \text{ Quits} &\quad + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(h_E\left(x,i,x'\right)\right) - V(x,i) \right] dF\left(x'\right) \\
\text{Incumbents} &\quad + \sum_{i=1}^{n(x)} \rho V_I(x,i) \\
\text{Firm} &\quad + \sum_{i=1}^{n(x)} \rho V_D(x,i)
\end{aligned}$$

where the indirect term due to incumbents can be written as:

$$\begin{aligned}
\sum_{i=1}^{n(x)} \rho V_I(x,i) &= \\
\text{Destructions} \quad & \sum_{i=1}^{n(x)} \sum_{j\neq i}^{n(x)} \delta \left[\mathbf{V}\left(d(x,j),i\right) - V\left(x,i\right)\right] \\
\text{Retentions} \quad & + \sum_{i=1}^{n(x)} \sum_{j\neq i}^{n(x)} \lambda^E \int_{x'\notin \mathcal{Q}^E(x,j)} \left[\mathbf{V}\left(r\left(x,j,x'\right),i\right) - V\left(x,i\right)\right] dF\left(x'\right) \\
EE \text{ Quits} \quad & + \sum_{i=1}^{n(x)} \sum_{j\neq i}^{n(x)} \lambda^E \int_{x'\in \mathcal{Q}^E(x,j)} \left[\mathbf{V}\left(q_E\left(x,j,x'\right),i\right) - V\left(x,i\right)\right] dF\left(x'\right)
\end{aligned}$$

and the indirect term due to the firm can be written as:

$$\begin{aligned}
\sum_{i=1}^{n(x)} \rho V_F(x,i) &= \\
UE \text{ Hires} \quad & \lambda^F v\left(x\right) \phi \sum_{i=1}^{n(x)} \left[\mathbf{V}\left(h_U\left(x\right),i\right) - V\left(x,i\right)\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE \text{ Threats} \quad & + \lambda^F v\left(x\right) \phi \sum_{i=1}^{n(x)} \left[\mathbf{V}\left(t_U\left(x\right),i\right) - V\left(x,i\right)\right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE \text{ Hires} \quad & + \lambda^F v\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x\in\mathcal{Q}^E(x',i')} \left[\mathbf{V}\left(h_E\left(x',i',x\right),i\right) - V\left(x,i\right)\right] dG\left(x',i'\right) \\
EE \text{ Threats} \quad & + \lambda^F v\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x\notin\mathcal{Q}^E(x',i')} \left[\mathbf{V}\left(t_E\left(x',i',x\right),i\right) - V\left(x,i\right)\right] dG\left(x',i'\right) \\
\text{Shocks} \quad & + \sum_{i=1}^{n(x)} \mathbb{E}_\eta \left[\mathbf{V}\left(g_z\left(x,\eta\right),i\right) - V\left(x,i\right)\right]
\end{aligned}$$

We now collect terms:

- **Destructions -** When worker i separates from firm x, the sum of the changes in values of all employed workers at its own firm is given by:

$$\begin{aligned}
\text{Destructions} &= \delta\left[U - V\left(x,i\right)\right] + \delta \sum_{j\neq i}^{n(x)} \left[\mathbf{V}\left(d(x,i),j\right) - V\left(x,j\right)\right] \\
&= \delta\left[U + \sum_{j\neq i}^{n(x)} \mathbf{V}\left(d(x,i),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right]
\end{aligned}$$

- **Retentions -** When i renegotiates at firm x, the sum of the changes in values of all employed workers at its own firm is given by:

$$\begin{aligned}
\text{Retentions} &= \lambda^E \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r\left(x,i,x'\right),i\right) - V\left(x,i\right)\right] dF\left(x'\right) + \lambda^E \int_{x' \notin \mathcal{Q}^E(x,i)} \sum_{j \neq i}^{n(x)} \left[\mathbf{V}\left(r\left(x,i,x'\right),j\right) - V\left(x,j\right.\right. \\
&= \lambda^E \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dF\left(x'\right) \\
&= \lambda^E \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\sum_{j=1}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dF\left(x'\right)
\end{aligned}$$

- **Quits** - Similarly, when i quits firm x, the sum of the changes in values of all employed workers at its own firm is given by:

$$EE \text{ Quits} = \lambda^E \int_{x' \in \mathcal{Q}(x,i)} \left[\mathbf{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \mathbf{V}\left(q_E\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dF\left(x'\right)$$

Summing up all these terms together, and defining for convenience :

$$\begin{aligned}
\rho \bar{V}\left(x\right) &= \sum_{i=1}^{n(x)} w\left(x,i\right) \\
\text{Destructions} \quad & + \sum_{i=1}^{n(x)} \delta \left[U + \sum_{j \neq i}^{n(x)} \mathbf{V}\left(d(x,i),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] \\
\text{Retentions} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\sum_{j=i}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dF\left(x'\right) \\
EE \text{ Quits} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \mathbf{V}\left(q_E\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dF\left(x'\right) \\
UE \text{ Hires} \quad & + \lambda^F v\left(x\right) \phi \sum_{i=1}^{n(x)} \left[\mathbf{V}\left(h_U\left(x\right),i\right) - V\left(x,i\right)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE \text{ Threats} \quad & + \lambda^F v\left(x\right) \phi \sum_{i=1}^{n(x)} \left[\mathbf{V}\left(t_U\left(x\right),i\right) - V\left(x,i\right)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE \text{ Hires} \quad & + \lambda^F v\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x \in \mathcal{Q}^E(x',i')} \left[\mathbf{V}\left(h_E\left(x',i',x\right),i\right) - V\left(x,i\right)\right] dG\left(x',i'\right) \\
EE \text{ Threats} \quad & + \lambda^F v\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x \notin \mathcal{Q}^E(x',i')} \left[\mathbf{V}\left(t_E\left(x',i',x\right),i\right) - V\left(x,i\right)\right] dG\left(x',i'\right) \\
\text{Shocks} \quad & + \mathbb{E}_\eta \sum_{i=1}^{n(x)} \left[\mathbf{V}\left(g_z\left(x,\eta\right),i\right) - V\left(x,i\right)\right]
\end{aligned}$$

Using the definition of $\bar{V}(x)$:

$$\begin{aligned}
\rho\bar{V}(x) &= \sum_{i=1}^{n(x)} w(x,i) \\
\text{Destructions} \quad & + \sum_{i=1}^{n(x)} \delta \left[U + \sum_{j\neq i}^{n(x)} \mathbf{V}(d(x,i),j) - \bar{V}(x) \right] \\
\text{Retentions} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x'\notin \mathcal{Q}^E(x,i)} \left[\sum_{j=i}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \bar{V}(x) \right] dF\left(x'\right) \\
EE \text{ Quits} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x'\in \mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \mathbf{V}\left(q_E\left(x,i,x'\right),j\right) - \bar{V}(x) \right] dF\left(x'\right) \\
UE \text{ Hires} \quad & + \lambda^F v(x)\,\phi \left[\sum_{i=1}^{n(x)} \mathbf{V}\left(h_U(x),i\right) - \bar{V}(x) \right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE \text{ Threats} \quad & + \lambda^F v(x)\,\phi \left[\sum_{i=1}^{n(x)} \mathbf{V}\left(t_U(x),i\right) - \bar{V}(x) \right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE \text{ Hires} \quad & + \lambda^F v(x)\,(1-\phi) \int_{x\in \mathcal{Q}^E(x',i')} \left[\sum_{i=1}^{n(x)} \mathbf{V}\left(h_E\left(x',i',x\right),i\right) - \bar{V}(x) \right] dG\left(x',i'\right) \\
EE \text{ Threats} \quad & + \lambda^F v(x)\,(1-\phi) \int_{x\notin \mathcal{Q}^E(x',i')} \left[\sum_{i=1}^{n(x)} \mathbf{V}\left(t_E\left(x',i',x\right),i\right) - \bar{V}(x) \right] dG\left(x',i'\right) \\
\text{Shocks} \quad & + \mathbb{E}_\eta \left[\sum_{i=1}^{n(x)} \mathbf{V}\left(g_z(x,\eta),i\right) \right] - \bar{V}(x)
\end{aligned}$$

Summing this last equation to the Bellman equation for $J(x)$ yields

$$
\begin{aligned}
\rho\Omega\left(x\right) \quad &= \quad y\left(x\right)-c\left(v\left(x\right),x\right) \\
\text{Destructions} \quad & \quad +\sum_{i=1}^{n(x)}\delta\left[\mathbf{J}\left(d(x,i)\right)+U+\sum_{j\neq i}^{n(x)}\mathbf{V}\left(d(x,i),j\right)-J\left(x\right)-\bar{V}\left(x\right)\right] \\
\text{Retentions} \quad & \quad +\lambda^{E}\sum_{i=1}^{n(x)}\int_{x'\notin\mathcal{Q}^{E}(x,i)}\left[\mathbf{J}\left(r\left(x,i,x'\right)\right)+\sum_{j=i}^{n(x)}\mathbf{V}\left(r\left(x,i,x'\right),j\right)-J\left(x\right)-\bar{V}\left(x\right)\right]dF\left(x'\right) \\
EE\text{ Quits} \quad & \quad +\lambda^{E}\sum_{i=1}^{n(x)}\int_{x'\in\mathcal{Q}^{E}(x,i)}\left[\mathbf{J}\left(q_{E}\left(x,i,x'\right)\right)+\mathbf{V}\left(h_{E}\left(x,i,x'\right),i\right)+\sum_{j\neq i}^{n(x)}\mathbf{V}\left(q_{E}\left(x,i,x'\right),j\right)-J\left(x\right)-\bar{V}\left(x\right)\right]dF \\
UE\text{ Hires} \quad & \quad +\lambda^{F}v\left(x\right)\phi\left[\mathbf{J}\left(h_{U}\left(x\right)\right)+\sum_{i=1}^{n(x)}\mathbf{V}\left(h_{U}\left(x\right),i\right)-J\left(x\right)-\bar{V}\left(x\right)\right]\cdot\mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\text{ Threats} \quad & \quad +\lambda^{F}v\left(x\right)\phi\left[\mathbf{J}\left(t_{U}\left(x\right)\right)+\sum_{i=1}^{n(x)}\mathbf{V}\left(t_{U}\left(x\right),i\right)-J\left(x\right)-\bar{V}\left(x\right)\right]\cdot\mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\text{ Hires} \quad & \quad +\lambda^{F}v\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^{E}(x',i')}\left[\mathbf{J}\left(h_{E}\left(x',i',x\right)\right)+\sum_{i=1}^{n(x)}\mathbf{V}\left(h_{E}\left(x',i',x\right),i\right)-J\left(x\right)-\bar{V}\left(x\right)\right]dG\left(x',i'\right) \\
EE\text{ Threats} \quad & \quad +\lambda^{F}v\left(x\right)\left(1-\phi\right)\int_{x\notin\mathcal{Q}^{E}(x',i')}\left[\mathbf{J}\left(t_{E}\left(x',i',x\right)\right)+\sum_{i=1}^{n(x)}\mathbf{V}\left(t_{E}\left(x',i',x\right),i\right)-J\left(x\right)-\bar{V}\left(x\right)\right]dG\left(x',i'\right) \\
\text{Shocks} \quad & \quad +\mathbb{E}_{\eta}\left[\mathbf{J}\left(g_{z}\left(x,\eta\right)\right)+\sum_{i=1}^{n(x)}\mathbf{V}\left(g_{\eta}\left(x,\eta\right),i\right)\right]-J\left(x\right)-\bar{V}\left(x\right)
\end{aligned}
$$

Collecting terms:

$$
\begin{aligned}
\rho\Omega\left(x\right) \quad &= \quad y\left(x\right)-c\left(v\left(x\right),x\right) \\
\text{Destructions} \quad & \quad +\sum_{i=1}^{n(x)}\delta\left[\mathbf{\Omega}\left(d(x,i)\right)+U-\Omega\left(x\right)\right] \\
\text{Retentions} \quad & \quad +\lambda^{E}\sum_{i=1}^{n(x)}\int_{x'\notin\mathcal{Q}^{E}(x,i)}\left[\mathbf{\Omega}\left(r\left(x,i,x'\right)\right)-\Omega\left(x\right)\right]dF\left(x'\right) \\
EE\text{ Quits} \quad & \quad +\lambda^{E}\sum_{i=1}^{n(x)}\int_{x'\in\mathcal{Q}^{E}(x,i)}\left[\mathbf{\Omega}\left(q_{E}\left(x,i,x'\right)\right)+\mathbf{V}\left(h_{E}\left(x,i,x'\right),i\right)-\Omega\left(x\right)\right]dF\left(x'\right) \\
UE\text{ Hires} \quad & \quad +\lambda^{F}v\left(x\right)\phi\left[\mathbf{\Omega}\left(h_{U}\left(x\right)\right)-U-\Omega\left(x\right)\right]\cdot\mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\text{ Threats} \quad & \quad +\lambda^{F}v\left(x\right)\phi\left[\mathbf{\Omega}\left(t_{U}\left(x\right)\right)-\Omega\left(x\right)\right]\cdot\mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\text{ Hires} \quad & \quad +\lambda^{F}v\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^{E}(x',i')}\left[\mathbf{\Omega}\left(h_{E}\left(x',i',x\right)\right)-\mathbf{V}\left(h_{E}\left(x',i',x\right),i'\right)-\Omega\left(x\right)\right]dG\left(x',i'\right) \\
EE\text{ Threats} \quad & \quad +\lambda^{F}v\left(x\right)\left(1-\phi\right)\int_{x\notin\mathcal{Q}^{E}(x',i')}\left[\mathbf{\Omega}\left(t_{E}\left(x',i',x\right)\right)-\Omega\left(x\right)\right]dG\left(x',i'\right) \\
\text{Shocks} \quad & \quad +\mathbb{E}_{\eta}\left[\mathbf{\Omega}\left(g_{z}\left(x,\eta\right)\right)-\Omega\left(x\right)\right]
\end{aligned}
$$

A.2 Proof of Conditions C1-C6

A.2.1 Proof of C1 and C2 (UE Hires and UE Threats)

- Consider a meeting between a firm x and an unemployed worker.
- Under **A-BP**, upon meeting an unemployed worker, internal renegotiation takes place first and external negotiation second, which is immediately followed by the worker joining the firm in the case of hiring. Finally, a second round of internal renegotiation may take place.
- We introduce some notation to keep track of values throughout the internal and external negotiations. To fix ideas, we denote by (IR1) the first round of internal negotiation, pre-external negotiation. We denote by (IR2) the second round of internal negotiation, post-hire.

  Post-hire and post-internal negotiation (IR2) values are denoted with double stars. Post-internal-negotiation (IR1) but pre-external-negotiation values are denoted with stars.

$$\Omega^{**} := J^{**} + \sum_{j=1}^{n(x)} V_j^{**} + V_i^{**}$$

$$\Omega^{*} := J^{*} + \sum_{j=1}^{n(x)} V_j^{*}$$

$$\Omega := J + \sum_{j=1}^{n(x)} V_j$$

- Proceeding by backward induction, under **A-BP** the firm makes a take-it-or-leave-it offer to the unemployed worker, therefore

$$V_i^{**} = U$$

- We now divide the proof in several steps. We start by proving two claims: (1) for all incumbent workers $j = 1...n(x)$, $V_j^{**} = V_j^{*}$, and (2) $\Omega^* = \Omega$. Once these claims have been proven, we move on to proving **C1 and Ce (UE Hires adn UE Threats)**.

Claim 1: For all incumbents workers $j = 1...n(x)$, we have $V_j^{**} = V_j^{*}$.

We proceed by backwards induction using our assumption **A-BP**.

- Immediately after (IR1) has taken place, only the following events can happen:

  1. Hire/not-hire

- Either the worker is hired from unemployment (H),
- Or the worker is not hired from unemployment (NH)

2. Possible new round of internal negotiation (IR2). This possible second round of internal negotiation (now including the newly hired worker) leads to values V_j^{**}.

- We focus on subgame perfect equilibria in this multi-stage game. Therefore, after (IR1), workers perfectly anticipate what the outcome of the hire/not-hire stage will be. That is, (IR1) is over, they know perfectly what hiring decision (H or NH) the firm will make.

- Now suppose that internal renegotatiation (IR2) actually happens after the hire/not-hire decision, that is, that for some incumbent worker $j \in \{1, ..., n(x)\}$, $V_j^{**} \neq V_j^*$. Note that the firm has no incentives to accept a change in the new worker's value to anything above U, so by **A-MC** her value does not change in the second round (IR2).

- We cnstruct the rest of the proof by contradiction. Suppose an incumbent worker j whose value changed in (IR2). Because of **A-MC**, her value can change only in the following cases:
  - The firm has a credible threat to fire worker j, in which case $V_j^{**} < V_j^*$
  - Worker j has a credible threat to quit, in which case $V_j^{**} > V_j^*$

- In addition, those credible threats can lead to a different outcome than in (IR1), and thus $V_j^{**} \neq V_j^*$, only if the threat on either side was not available in (IR1). If that same threat was available in the first round (IR1), then the outcome of the bargaining (IR1) would have been V_j^{**}.

- Recall that both incumbent worker j and the firm understand and anticipate which hire/not-hire decision the firm will make after the first round (IR1). They also understand and anticipate that, in case of hire, the value of the new worker will remain U in the second round (IR2).

- Therefore, the firm can *credibly threaten* to hire the new worker in the first round *if and only if* it actually hires her after the first round (IR1) is over. This implies that the firm can credibly threaten worker to fire j in the second round (IR2), by **A-LC**, *if and only if* it could credibly threaten her with hiring the new worker *in the first round of internal renegotiation (IR1).* This in turn entails that any credible threat the firm can make in the second round (IR2), it could already make in the first round (IR1).

- On the worker side, clearly quitting into unemployment is a credible threat when her value is below the value of unemployment. So this threat does not change between the first round (IR1) and the second round (IR2), because the equilibrium value to that worker will always be above the value of unemployment.

- In sum, the set of credible threats both to the firm and to worker j does not change between the initial round of internal renegotiation (IR1) and the post-hiring-decision round (IR2).

- This finally implies that the outcome of the initial round of internal renegotiation (IR1) for any incumbent j remains unchanged in the second round (IR2), that is:

$$V_j^{**} = V_j^*$$

which proves **Claim 1**. We now turn to the proof of **Claim 2**.

Claim 2: $\Omega^* = \Omega$.

Here we prove our earlier statement that *"the joint value stays constant before and after an internal negotiation"*. The logic is similar (but not identical) to the proof for vacancies. We proceed by contradiction.

1. Suppose that $\Omega^* < \Omega$. This inequality implies that either the firm or one of the worker loses. The logic here is to show that the losing parties would make transfers to the winning to go back to the old value structure. We study both cases separately.

   (a) Suppose that $J^* < J$.

   - Because of **A-MC**, the firm can lose value only if a worker has a credible threat to quit into unemployment. However, the fact that the firm has met with an unemployed worker does not change whether that threat is credible for any incumbent worker.
   - Therefore, the worker would have triggered renegotiation before the meeting, and $J^* \geq J$, our contradiction.

   (b) Since the firm cannot lose from the meeting, there must exist a non-empty set of incumbent workers $\mathcal{J} \subseteq \{1, ..., n(x)\}$ such that for all $j \in \mathcal{J}$, $V_j^* < V_j$.

   - Because of $\Omega^* < \Omega$ and of assumption **A-MC** on the firm side, all workers $j \notin \mathcal{J}$ have constant value $V_j^* = V_j$.

- We now write

$$\begin{aligned}\Omega - \Omega^* &= J + \sum_{j=1}^{n(x)} V_j - J^* - \sum_{j=1}^{n(x)} V_j^* \\ &= [J - J^*] + \sum_{j \in \mathcal{J}} [V_j - V_j^*]\end{aligned}$$

Therefore

$$\underbrace{[\Omega - \Omega^*]}_{>0} + \underbrace{[J^* - J]}_{\geq 0} = \sum_{j \in \mathcal{J}} [V_j - V_j^*]$$

- We first show that there is a collective, profitable deviation. We then show that the same deviation can be achieved by a single worker.

   i. (Collective deviation)

   – Consider the following deviation.:
   Suppose all workers $j \in \mathcal{J}$ collectively transfer $J^* - J$ to the firm in exchange for getting back their old values at V_j. The fact that workers can request that the firm sets back their values in exchange for the transfer is possible because of assumption **A-TR**.

   – After the transfers and the new values are set, the joint value is back to Ω. Similarly, the value of the firm is back to J, plus any transfers it receives under the deviation.

   – The firm ends up with

   $$J + [J^* - J] = J^*$$

   and is thus indifferent.

   – Workers $j \notin \mathcal{J}$ are also indifferent, because for them nothing changes.

   – The workers in $\mathcal{J}$ are (collectively) strictly better off, since they get a collective value increase of

   $$\sum_{j \in \mathcal{J}} [V_j - V_j^*] - [J^* - J] = \Omega - \Omega^* > 0$$

   – Therefore, provided workers can decide collectively on the transfer, they decide to make the transfer that restores Ω as the coalition value. This is our contradiction.

   ii. (Individual deviation)

   – Consider the following deviation:

One worker $j_0 \in \mathcal{J}$ proposes the following deviation that: (i) she transfers $J^* - J - \sum_{j \in \mathcal{J} \setminus \{j_0\}} [V_j - V_j^*]$ to the firm (possible because of assumption **A-TR**), in exchange of which (ii) she requests that the firm sets back the values of workers $j \in \mathcal{J}$ back to V_j, including herself, (iii) the firm asks each worker $j \in \mathcal{J} \setminus \{j_0\}$ for a transfer $V_j - V_j^*$.

– Under this deviation, the firm ends up with

$$J + \underbrace{[J^* - J] - \sum_{j \in \mathcal{J} \setminus \{j_0\}} [V_j - V_j^*]}_{\text{transfer from } j_0} + \underbrace{\sum_{j \in \mathcal{J} \setminus \{j_0\}} [V_j - V_j^*]}_{\text{transfers from all } j \in \mathcal{J} \setminus \{j_0\}} = J^*$$

and is thus indifferent.

– All workers $j \in \mathcal{J} \setminus \{j_0\}$ get

$$V_j - (V_j - V_j^*) = V_j^*$$

and are thus indifferent.

– Finally, worker j_0 gets a change in value of

$$\begin{aligned} V_{j_0} - V_{j_0}^* - \left\{ [J^* - J] - \sum_{j \in \mathcal{J} \setminus \{j_0\}} [V_j - V_j^*] \right\} &= \sum_{j \in \mathcal{J}} [V_j - V_j^*] - [J^* - J] \\ &= \Omega - \Omega^* > 0 \end{aligned}$$

and is thus strictly better off. Hence, we have shown that a single worker can construct a system of transfers that (i) makes all individuals indifferent and herself strictly better off, and (ii) restores the joint value Ω.

- Therefore, any worker would implement that system of transfers after the first round of internal renegotiation (IR1) when $\Omega^* < \Omega$, and the joint value would be back to Ω: we have obtained our contradiction.

2. Now suppose for a contradiction that $\Omega^* > \Omega$. This inequality implies that either the firm, either some workers gain. The logic here is to show that in this case renegotiation would have happened earlier. Because the argument are very similar to the case $\Omega^* < \Omega$, we only provided a sketch of them.

   (a) Suppose that no agent loses, i.e. all workers and the firm are weakly better off: $J^* \geq J$ and $V_j^* \geq V_j$, with at least one agent strictly better off. In this case the agent strictly better off would have had triggered renegotiation before the new meeting, by offering J^*, V^* to all other parties, and a value increase of $\Omega^* - \Omega$ to itself. Under this deviation, all parties are weakly better off, and the agent under consideration is strictly better off.

(b) Suppose that at least one agent loses, for instance worker j_0. In this case the winning parties would have triggered renegotiation earlier, by offering to the losing agent a transfer $V_{j_0} - V_{j_0}^*$ in exchange for downgrading value to $V_{j_0}^*$. The winning parties would have then pocketed the difference $\Omega^* - \Omega > 0$.

In sum, we have show our earlier statement that *"the joint value stays constant before and after an internal negotiation"*, namely:

$$\Omega^* = \Omega$$

which is equivalent to

$$J^* + \sum_{j=1}^{n(x)} V_j^* = J + \sum_{j=1}^{n(x)} V_j$$

We can now use **Claim 1** $V_j^{**} = V_j^*$ and **Claim 2** $\Omega^* = \Omega$.

- Using the definitions of Ω^{**} and Ω, we can write

$$\Omega^{**} - \Omega = \left[J^{**} + \sum_{j=1}^{n(x)} V_j^{**} + V_i^{**}\right] - \left[J + \sum_{j=1}^{n(x)} V_j\right]$$

- Now using $V_i^{**} = U$, we obtain

$$\Omega^{**} - \Omega = \left[J^{**} + \sum_{j=1}^{n(x)} V_j^{**}\right] - \left[J + \sum_{j=1}^{n(x)} V_j\right] + U$$

- Using **Claim 1:** $V_j^{**} = V_j^*$, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - \left[J^* + \sum_{j=1}^{n(x)} V_j^*\right] - \left[J + \sum_{j=1}^{n(x)} V_j\right] + U$$

- Finally using **Claim 2:** $\Omega^* = \Omega$, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] + U$$

which can be re-written

$$J^{**} - J^* = [\Omega^{**} - \Omega] - U$$

- Now under **A-LC**, the firm will only hire if its value after hiting is higher than its value after internal renegotiation: $J^{**} - J^{*} \geq 0$. This inequality requires

$$\Omega^{**} - \Omega \geq U$$
$$\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) \geq U$$

- The firm does not hire when its value of hiring is below its value of renegotiation $J^{**} < J^{*}$. This inequality implies

$$\Omega^{**} - \Omega < U$$

- When the firm does not hire, we obtain using **Claim 2**:

$$\Omega^{**} - \Omega^{*} < U$$

  which finally implies

$$\mathbf{\Omega}\left(h_U\left(x\right)\right) - \mathbf{\Omega}\left(t_U(x)\right) < U$$

Now, we argue the following.

Claim 3 Conditional on not hiring, $\Omega^{**} = \Omega^{*} = \Omega$, where in this case Ω^{**} denotes the value of the coalition without hiring, and thus does not include the value of the unemployed worker.

- To show **Claim 3**, apply the exact same reasoning as for **Claim 2**, under the assumption that the firm does not hire.

- Now, by **Claim 3**, $\Omega^{**} = \Omega^{*} = \Omega$ when there is no hiring. Therefore:

$$\mathbf{\Omega}\left(t_U(x)\right) = \Omega(x)$$

- We have therefore shown **C1 and C2 (*UE* Hires and *UE* Threats)**: An unemployed worker that meets x is hired when$x \in \mathcal{Q}^U$, where

$$\mathcal{A} = \left\{x \middle| \mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) \geq U\right\}$$

  and upon joining the firm, has value

$$\mathbf{V}\left(h_U\left(x, i\right)\right) = U.$$

and

$$\boldsymbol{\Omega}(t_U(x)) = \Omega(x)$$

A.2.2 Proof of C1 and C3 (EE Hires, EE Threats and Retentions)

- Consider firm x that has met worker i' at firm x'. We seek to determine $\mathcal{Q}^E(x', i')$.
- Under **A-BP**, upon meeting an employed worker, internal negotiation first takes place at the poaching firm (x), then x makes a take-it-or-leave-it offer. Internal negotiation then takes place at (x') with all workers including i'.
- Proceeding by backward induction, we again define intermediate values but here at x', noting that $q_E(x', i', x)$ gives the number of employees in x' if the worker leaves:

$$\Omega = J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}$$

$$\Omega^* = J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^* + V_{i'}^*$$

$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**}$$

- Note, in the second line we are describing the values of the firm in renegotiation where i' stays with the firm, so $V_{i'}^*$ is the outcome of internal negotiation. In the third line we consider the firm having lost the worker. Under **A-BP** the firm will respond to an offer $\bar{V}$ from x with

$$V_{i'}^* = \bar{V}$$

- The same result as in **Claim 1** from section A.2.1 obtains: under **A-TR** and **A-BP**, the values accepted by the incumbent workers *after the internal renegotiation* $\left(V_j^*\right)$ will be equal to the values they receive *after the external negotiation* $\left(V_j^{**}\right)$, that is

$$V_j^{**} = V_j^*$$

The argument would be exactly the same.

- Using these two results and the above definitions

$$\Omega^{**} - \Omega = \left[J^{**} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**}\right] - \left[J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}\right]$$

$$\Omega^{**} - \Omega = \left[J^{**} + J^{*} - J^{*} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**} + V_{i'}^{*} - V_{i'}^{*}\right] - \left[J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}\right]$$

$$\Omega^{**} - \Omega = [J^{**} - J^{*}] + \left[J^{*} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{*} + V_{i'}^{*}\right] - \left[J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}\right] - V_{i'}^{*}$$

$$\Omega^{**} - \Omega = [J^{**} - J^{*}] + [\Omega^{*} - \Omega] - V_{i'}^{*}$$

$$\Omega^{**} - \Omega = [J^{**} - J^{*}] + [\Omega^{*} - \Omega] - \bar{V}$$

- The same result as in **Claim 2** from section A.2.1 obtains: because of **A-TR**, any value lost to the firm must accrue to its workers, while any value lost to a worker must accrue either to the firm, or to another worker, which we earlier formulated as *"the joint value stays constant before and after an internal negotiation"*. Mathematically, this statement translates into

$$\Omega^{*} = \Omega$$

- Subsituting into the equation that we obtained above $\Omega^{**} - \Omega = [J^{**} - J^{*}] + [\Omega^{*} - \Omega] - \bar{V}$, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^{*}] - \bar{V}$$

- Now under **A-LC**, the firm x' will only try to keep the worker if $J^{*} > J^{**}$, which requires

$$\Omega - \Omega^{**} \leq \bar{V}$$

$$\mathbf{\Omega}\left(r(x',i',x) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right) \leq \bar{V}$$

- This determined the maximum value that x' can offer to the worker to retain them. Knowing that firm x' can counter at most with $\bar{V} = \mathbf{\Omega}\left(r(x',i',x) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)$, then will firm x succesfully poach the worker?

- First, note that the bargaining protocol implies that x firm will offer $\bar{V}$ if it is making an offer, since it need not offer more.

- For firm x the argument may proceed identically to the case of unemployment, simply replacing U

with $\bar{V}$. The result is that the firm will hire only if

$$\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega\left(x\right)\geq\bar{V}$$

or

$$\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega\left(x\right)\geq\boldsymbol{\Omega}\left(r(x',i',x)\right)-\boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)$$

- Finally, when firm x does not hire, the same argument as in **Claim 3** in Section A.2.1 applies: $\Omega^{**}=\Omega^{*}=\Omega$. This observation implies

$$\boldsymbol{\Omega}(t_E(x',i',x))=\Omega(x)$$

Similarly, the same argument as in **Claim 3** implies that when firm x' does not lose its worker, $\Omega^{**}=\Omega^{*}=\Omega$, thereby implying

$$\boldsymbol{\Omega}(r(x',i',x))=\Omega(x')$$

- The combination of these conditions deliver **C1 and C2 (*EE* Hires, *EE* Threats and Retention)**

1. The quit set of an employed worker is determined by

$$\mathcal{Q}^E\left(x',i'\right)=\left\{x\middle|\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega\left(x\right)\geq\Omega\left(x'\right)-\boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)\right\}$$

2. The worker's value of being hired from employment from firm x' is

$$\mathbf{V}(h_E(x,x',i'))=\Omega\left(x'\right)-\boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)$$

3. Worker i''s value of being retained at x' after meeting x is[21]

$$\mathbf{V}(r(x',i',x),i')=\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega\left(x\right)$$

4. The joint value of the potential poaching firm x when the worker is not hired does not change:

$$\boldsymbol{\Omega}(t_E(x',i',x))=\Omega(x)$$

[21] Because offers are made at no cost, both firms always make an offer, even when they know that they cannot retain/hire the worker in equilibrium. This is exactly the same as in Postel-Vinay Robin (2002).

5. The joint value of the potential poached firm x' does not change when the worker stays:

$$\boldsymbol{\Omega}(r(x', i', x)) = \Omega(x')$$

A.2.3 Proof of C4 (EU Quits)

We first show that

$$\mathcal{L} = \left\{ (x, i) \middle| \Omega\left(s\left(x, \left(1 - \left[\ell\left(x\right); \ell_i\left(x\right) = 1\right]\right) \circ \left(1 - q_U\left(x\right)\right)\right), i\right) + U > \Omega\left(s\left(x, \left(1 - \left[\ell\left(x\right); \ell_i\left(x\right) = 0\right]\right) \circ \left(1 - q_U\left(x\right)\right)\right), \right.$$

from the firm side, then that

$$\mathcal{Q}^U = \left\{ (x, i) \middle| \Omega\left(s\left(x, \left(1 - \ell\left(x\right)\right) \circ \left(1 - \left[q_{U,-i}\left(x\right); q_{U,i}\left(x\right) = 1\right]\right)\right), i\right) + U > \Omega\left(s\left(x, \left(1 - \ell\left(x\right)\right) \circ \left(1 - \left[q_{U,-i}\left(x\right); q_{U,i}\left(x\right.\right.\right.\right.\right.$$

on the worker side.

Part 1: Firm side

$$\mathcal{L} = \left\{ (x, i) \middle| \Omega\left(s\left(x, \left(1 - \left[\ell\left(x\right); \ell_i\left(x\right) = 1\right]\right) \circ \left(1 - q_U\left(x\right)\right)\right), i\right) + U > \Omega\left(s\left(x, \left(1 - \left[\ell\left(x\right); \ell_i\left(x\right) = 0\right]\right) \circ \left(1 - q_U\left(x\right)\right)\right), \right.$$

- Consider a firm x who is considering laying off worker i for whom $q_{U,i}(x) = 0$. As above, we start with definitions, noting that $n\left(s\left(x, \left(1 - \left[\ell\left(x\right); \ell_i\left(x\right) = 1\right]\right) \circ \left(1 - q_U\left(x\right)\right)\right)\right)$ is the number of workers if i is laid off.

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i$$
$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*$$
$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}$$

- Note that in the first line the coalition has still worker i in it. In the second line, the firm and the worker i have negotiated (and internal negotiation has determined V_i^* which is what i will get if they stay in the firm). In the third line, the worker has been fired and another round of negotiation has occurred among incumbents.

- The same result as in **Claim 1** from section A.2.1 obtains: under **A-TR** and **A-BP**, the values accepted by the incumbent workers *after the internal renegotiation* $\left(V_j^*\right)$ will be equal to the values they receive *after the external negotiation* $\left(V_j^{**}\right)$, that is

$$V_j^{**} = V_j^*$$

The argument would be exactly the same.

- Using this result and the above definitions

$$\begin{aligned}
\Omega^{**} - \Omega &= \left[J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= \left[J^{**} - J^* + J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^* - V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] - V_i^* \\
\Omega^{**} - \Omega &= [J^{**} - J^*] + [\Omega^* - \Omega] - V_i^*
\end{aligned}$$

- The same result as in **Claim 2** from section A.2.1 obtains: because of **A-TR**, any value lost to the firm must accrue to its workers, while any value lost to a worker must accrue either to the firm, or to another worker, which we earlier formulated as *"the joint value stays constant before and after an internal negotiation"*. Mathematically, this statement translates into

$$\Omega^* = \Omega$$

- Substituting this equation into the one above, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - V_i^*$$

- Now under **(A-LC)** the firm x will only layoff the worker if $J^{**} > J^*$, which requires

$$\Omega - \Omega^{**} < V_i^*$$

- As long as $V_i^* > U$ the worker would be willing to transfer value to the firm to avoid being laid off, implying

$$\Omega - \Omega^{**} < U.$$

which we can re-write

$$\Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=1\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),i\right)+U > \Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=0\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),i\right)$$

where the LHS is $\Omega^{**} + U$ (under the layoff) and the RHS is Ω. This concludes the proof for the firm side.

Part 2: Worker side

$$\mathcal{Q}^U = \Big\{ (x,i) \Big| \Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=1\right]\right)\right),i\right) + U > \Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(\right.\right.\right.\right.\right.$$

- Consider worker i in firm x who is considering quitting to unemployment for whom $\ell_i(x) = 0$. As above, we start with definitions, noting that $n\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=1\right]\right)\right)\right)$ is the number of workers if i quits. As before,

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i$$

$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*$$

$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}$$

- The same result as in **Claim 1** from section A.2.1 obtains: under **(A-TR)** and **(A-BP)**, the values accepted by the incumbent workers *after the internal renegotiation* $\left(V_j^*\right)$ will be equal to the values they receive *after the external negotiation* $\left(V_j^{**}\right)$, that is

$$V_j^{**} = V_j^*$$

  The argument would be exactly the same.

- Using this result and the above definitions

$$\begin{aligned}
\Omega^{**} - \Omega &= \left[J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= \left[J^{**} + J^* - J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^* - V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= \left[J^{**} - J^*\right] + \left[J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] - V_i^* \\
\Omega^{**} - \Omega &= \left[J^{**} - J^*\right] + \left[\Omega^* - \Omega\right] - V_i^*
\end{aligned}$$

- The same result as in **Claim 2** from section A.2.1 obtains: because of **A-TR**, any value lost to the firm must accrue to its workers, while any value lost to a worker must accrue either to the firm, or to another worker, which we earlier formulated as *"the joint value stays constant before and after an internal negotiation"*. Mathematically, this statement translates into

$$\Omega^* = \Omega$$

- Substituting this equation into the one above, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - V_i^*$$

- Now under **(A-LC)**, worker i will quit into unemployment iff $V_i^* < U$, which requires

$$J^{**} - J^* + [\Omega - \Omega^{**}] < U$$

As long as $J^{**} < J^*$, the firm is willing to transfer value to worker i to retain her. Therefore, worker i quits into unemployment iff the previous inequality holds at $J^{**} = J^*$, i.e.

$$\Omega - \Omega^{**} < U$$

- Therefore, the worker quits iff

$$\Omega\left(s\left(x, \left(1-\ell\left(x\right)\right) \circ \left(1-\left[q_{U,-i}\left(x\right); q_{U,i}\left(x\right)=1\right]\right)\right), i\right) + U > \Omega\left(s\left(x, \left(1-\ell\left(x\right)\right) \circ \left(1-\left[q_{U,-i}\left(x\right); q_{U,i}\left(x\right)=0\right]\right)\right)\right)$$

which concludes the proof of the worker side.

- This delivers $C4$.

A.2.4 Proof of C5 (Exit)

- Consider a firm x who contemplates exit after all endogenous quits and layoffs, thus when its employment is $n\left(s\left(x, \kappa\left(x\right)\right)\right)$
- As before we define values conditional on exiting:

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j$$

$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^*$$

$$\Omega^{**} = J^{**} + 0$$

Notice that the joint value after exit is simply the value of the firm, since all other workers have left because of exit.

- We can compute:

$$\begin{aligned}
\Omega^{**} - \Omega &= J^{**} - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j \right] \\
\text{(add and subtract } J^*) &= [J^{**} - J^*] + J^* - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j \right] \\
\text{(add and substract } \sum_{j=1}^{n(s(\cdot))} V_j^*) &= [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* \right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j \right] - \sum_{j=1}^{n(s(\cdot))} V_j^* \\
\text{(definition of } \Omega, \Omega^*) &= [J^{**} - J^*] + [\Omega^* - \Omega] - \sum_{j=1}^{n(s(\cdot))} V_j^*
\end{aligned}$$

- The same result as in **Claim 2** from section A.2.1 obtains: because of **A-TR**, any value lost to the firm must accrue to its workers, while any value lost to a worker must accrue either to the firm, or to another worker, which we earlier formulated as *"the joint value stays constant before and after an internal negotiation"*. Mathematically, this statement translates into

$$\Omega^* = \Omega$$

- Therefore, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - \sum_{j=1}^{n(s(\cdot))} V_j^*$$

- The firm exits iff $J^{**} \geq J^*$, that is, $\vartheta \geq J^*$. This is equivalent to

$$\Omega^{**} - \Omega \geq - \sum_{j=1}^{n(s(\cdot))} V_j^*$$

- Using again that $\Omega^{**} = J^{**} = \vartheta$, the firm exits iff

$$\vartheta + \sum_{j=1}^{n(s(\cdot))} V_j^* \geq \Omega$$

- Since any worker is better off under $V_i^* \geq U$ than unemployed of the firm exits, all workers are willing to take a value cut down to U if $\vartheta - \Omega + \sum_{j=1}^{n(s(\cdot))} V_j^* \geq 0$ because then the firm can credibly exit.

- This implies that the firm exits if and only if

$$\vartheta - \Omega\left(s\left(x, \kappa\left(x\right)\right)\right) + n\left(s\left(x, \kappa\left(x\right)\right)\right) U \geq 0$$

- This proves **C5 (Exit)**: the set of x such that the firm exits is given by

$$\mathcal{E} = \left\{ x \middle| \vartheta + n\left(s\left(x, \kappa\left(x\right)\right)\right) \cdot U \geq \Omega\left(s\left(x, \kappa\left(x\right)\right)\right) \right\}$$

A.2.5 Proof of C6 (Vacancies)

We split the proof in two steps. First, we show that workers are collectively willing to transfer value to the firm in exchange for the joint value-maximizing vacancy policy function. Second, we show that a single worker can create a system of transfers that achieves the same outcome.

Part 1: Collective transfers In this step, we show that workers are collectively better off transfering value to the firm in exchange of the firm posting the joint value-maximizing amount of vacancies.

- The vacancy posting decision that maximizes firm value is:

$$\frac{c_v\left(v\left(x\right), n\left(x\right)\right)}{\lambda^F} = \phi\left[\mathbf{J}\left(h_U\left(x\right)\right) - J\left(x\right)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} + (1-\phi) \int_{x \in \mathcal{Q}^E(x', i')} \left[\mathbf{J}\left(h_E\left(x', i', x\right)\right) - J\left(x\right)\right] dG\left(x', i'\right).$$

- Coherently with the rest of the analysis, we allow transfers between workers and firm that can affect the vacancy creation decision.

- Let v^J be the policy that maximizes the value of the firm, v^Ω be the policy that maximizes the value of the coalition, and $v^{\bar{V}}$ be the policy that maximizes the value of all the employees.

- Let $\Omega^\gamma, J^\gamma, \bar{V}^\gamma$ be the value of the coalition, firm and all workers under the v^γ, for $\gamma \in \{\Omega, J, \bar{V}\}$.

- **Claim (a): Collective value gains -** Under assumption **A-TR**, the policy v^Ω will lead to $\bar{V}^\Omega \geq \bar{V}^J + [J^J - J^\Omega]$ where $J^J - J^\Omega \geq 0$.

  - By construction Ω^Ω is greater than Ω^J.

  $$\Omega^\Omega \geq \Omega^J$$

  - By definition: $\Omega^\Omega = J^\Omega + \bar{V}^\Omega$, and $\Omega^J = J^J + \bar{V}^J$, therefore

  $$\begin{aligned} J^\Omega + \bar{V}^\Omega &\geq J^J + \bar{V}^J \\ \bar{V}^\Omega - \bar{V}^J &\geq J^J - J^\Omega \end{aligned}$$

- Since J^J is the value under the optimal policy for J, then $J^J \geq J^\Omega$. The above then implies that

$$\bar{V}^\Omega - \bar{V}^J \geq J^J - J^\Omega \geq 0$$

- This implies that workers would be prepared to transfer $T = J^J - J^\Omega$ to the firm in order for the firm to pursue policy v^Ω instead of v^J.

- **Claim (b): Infeasibility of $\bar{V}^{\bar{V}}$ -** There does not exist an incentive-compatible transfer from workers to firm that will lead to $\bar{V}^{\bar{V}}$:

  - Suppose workers consider transfering even more to induce the firm to follow policy $v^{\bar{V}}$ that maximizes their value.
  - By construction $\Omega^\Omega \geq \Omega^{\bar{V}}$. Using definitions for each of these, then $J^\Omega + \bar{V}^\Omega \geq J^{\bar{V}} + \bar{V}^{\bar{V}}$. Rearranging this:

$$J^\Omega - J^{\bar{V}} \geq \bar{V}^{\bar{V}} - \bar{V}^\Omega$$

  - Since $\bar{V}^{\bar{V}}$ is the value under the optimal policy for $\bar{V}$, then $\bar{V}^{\bar{V}} \geq \bar{V}^\Omega$. The above then implies that

$$J^\Omega - J^{\bar{V}} \geq \bar{V}^{\bar{V}} - \bar{V}^\Omega \geq 0$$

  - Taking v^Ω as a baseline, the above implies that a change to $v^{\bar{V}}$ causes a loss of $J^\Omega - J^{\bar{V}}$ to the firm, which is more than the gain of $\bar{V}^{\bar{V}} - \bar{V}^\Omega$ to the workers. The workers could transfer all of their gains under $v^{\bar{V}}$ to the firm, but the firm would still not choose $v^{\bar{V}}$ over v^Ω. This concludes the proof of **Claim (b)**

- **Claim (c): Optimality of $\bar{V}^\Omega$ -** There does not exist an incentive-compatible transfer from workers to firm that will lead to $\bar{V}^* \in \left(\bar{V}^\Omega, \bar{V}^{\bar{V}}\right)$

  - Call such a policy $v^{\bar{V}*}$. Then: $\Omega^\Omega \geq \Omega^{\bar{V}*}$, and by definitions

$$\begin{aligned} J^\Omega + \bar{V}^\Omega &\geq J^{\bar{V}^*} + \bar{V}^{\bar{V}^*} \\ J^\Omega - J^{\bar{V}^*} &\geq \bar{V}^{\bar{V}^*} - \bar{V}^\Omega \end{aligned}$$

  - Since by definition $\bar{V}^* \in \left(\bar{V}^\Omega, \bar{V}^{\bar{V}}\right)$, then $\bar{V}^{\bar{V}^*} - \bar{V}^\Omega \geq 0$. Therefore

$$J^\Omega - J^{\bar{V}^*} \geq \bar{V}^{\bar{V}^*} - \bar{V}^\Omega \geq 0$$

- Taking v^{Ω} as a baseline, the above implies that a change to $v^{\bar{V}^*}$ causes a loss of $J^{\Omega} - J^{\bar{V}^*}$ to the firm, which is more than the gain of $\bar{V}^{\bar{V}^*} - \bar{V}^{\Omega}$ to the workers.

- In summary, it is optimal for workers to transfer exactly $T = J^J - J^{\Omega}$ to the firm, in order for the firm to pursue v^{Ω} instead of v^J. Further transfers to the firm would be required to have the firm pursue a better policy for workers, but this is exceedingly costly to the firm and the workers are unwilling to make a transfer to cover these costs. This concludes the proof of **Step 1: Collective transfers.**

Part 2: Individual transfers In this step, we show that a single worker can construct a system of transfers that induces the firm to post v^{Ω} instead of v^J, while leaving all agents better off.

- Suppose no worker makes a transfer to the firm.
- Suppose worker j_0 deviates as follows. Worker j_0 makes a transfer $J^J - J^{\Omega}$ to the firm, in exchange of what (i) the firm posts v^{Ω} instead of v^J, and (ii) the worker gets a wage increase that gives her all the differential surplus $\bar{V}^{\Omega} - \bar{V}^J$.
- Following the same steps as in Part 1: Colelctive transfers, the firm gets $J^{\Omega} + [J^J - J^{\Omega}] = J^J$ and is hence indifferent.
- Similarly, workers $j \neq j_0$ do not get any value change, and are thus indifferent
- Finally, worker j_0 gets a value increase of

$$[\bar{V}^{\Omega} - \bar{V}^J] - [J^J - J^{\Omega}] \geq 0$$

  where the inequality follows from **Part 1: Collective transfers**.

- This argument shows that a single worker has an incentive to and can induce the firm to post v^{Ω}.
- Notice also that the same argument holds starting from any vacancy policy function $\tilde{v} \neq v^J$ together with a falue of the firm $\tilde{J}$. Thus, even if some worker induces the firm to post a different vacacy policy function which is not v^{Ω} any other worker has an incentive to induce the firm to post v^{Ω}
- Therefore, in equilibrium, the firm posts v^{Ω}

A.3 Applying Conditions C1-C6

Having established that **Assumption A1-A6** can be used to prove **Conditions C1-C6**, we now apply **C1-C6** to the total value Bellman equation. We proceed one term at the time, working through (i) exogenous destructions, (ii) retentions, (iii) EE (poached) quits, (iv) UE hires, (v) UE threats, (vi) EE (poached) hires, (vii) EE threats, and (viii) productivity shocks.

(i) Exogenous destructions

$$\begin{aligned}
\text{Destructions} &= \sum_{i=1}^{n(x)} \delta \left[\mathbf{J}\left(d(x,i)\right) + U + \sum_{j \neq i}^{n(x)} \mathbf{V}\left(d(x,i), j\right) - \Omega\left(x\right) \right] \\
&= \sum_{i=1}^{n(x)} \delta \left[\mathbf{J}\left(d(x,i)\right) + \sum_{j=1}^{n(d(x,i))} \mathbf{V}\left(d(x,i), j\right) + U - \Omega\left(x\right) \right] \\
&= \sum_{i=1}^{n(x)} \delta \left[\mathbf{\Omega}\left(d(x,i)\right) + U - \Omega\left(x\right) \right] \\
\text{Destructions} &= -\delta \sum_{i=1}^{n(x)} \left[\Omega\left(x\right) - \mathbf{\Omega}\left(d(x,i)\right) - U \right]
\end{aligned}$$

where we simply have used the definition of $\Omega\left(d(x,i)\right) := \mathbf{J}\left(d(x,i)\right) + \sum_{j=1}^{n(d(x,i))} \mathbf{V}\left(d(x,i), j\right)$.

(ii) Retentions

$$\begin{aligned}
\text{Retentions} &= \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\mathbf{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right), j\right) - \Omega\left(x\right) \right] dF\left(x'\right) \\
&= \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\mathbf{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega\left(x\right) \right] dF\left(x'\right)
\end{aligned}$$

where we simply have used the definition of $\mathbf{\Omega}\left(r\left(x,i,x'\right)\right) := \mathbf{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right), j\right)$. Now using the result in **C1** that

$$\mathbf{\Omega}\left(r\left(x,i,x'\right)\right) = \Omega(x')$$

we obtain that

$$\text{Retentions} = 0$$

(iii) EE **Quits**

$$EE \text{ Quits} = \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\mathbf{J}\left(q_E\left(x,i,x'\right)\right) + \mathbf{V}\left(q_E\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \mathbf{V}\left(q_E\left(x,i,x'\right),j\right) - \Omega\left(x\right) \right] dF\left(x'\right)$$

Now by definition

$$\boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right) = \mathbf{J}\left(q_E\left(x,i,x'\right)\right) + \sum_{j=1}^{n(q_E(x,i,x'))} \mathbf{V}\left(q_E\left(x,i,x'\right),j\right)$$

$$\boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right) = \mathbf{J}\left(q_E\left(x,i,x'\right)\right) + \sum_{j \neq i}^{n(x)} \mathbf{V}\left(q_E\left(x,i,x'\right),j\right)$$

$$\mathbf{J}\left(q_E\left(x,i,x'\right)\right) + \sum_{j \neq i}^{n(x)} \mathbf{V}\left(q_E\left(x,i,x'\right),j\right) = \boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right)$$

Using this in the term in square brackets

$$EE \text{ Quits} = \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\Omega\left(q_E\left(x,i,x'\right)\right) - \Omega\left(x\right) + \mathbf{V}\left(q_E\left(x,i,x'\right),i\right)\right] dF\left(x'\right)$$

Under **C3**, the value going to the poached worker is $\mathbf{V}\left(q_E\left(x,i,x'\right)\right) = \Omega\left(x\right) - \Omega\left(q_E\left(x,i,x'\right)\right)$, using this, the term in the square brackets is zero, so

$$EE \text{ Quits} = 0$$

(iv) UE **Hires**

$$UE \text{ Hires} = \lambda^F v\left(x\right) \phi \left[\mathbf{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \mathbf{V}\left(h_U\left(x\right),i\right) - \Omega\left(x\right)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}$$

Now by definition

$$\boldsymbol{\Omega}\left(h_U\left(x\right)\right) = \mathbf{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(h_U(x))} \mathbf{V}\left(h_U\left(x\right),i\right)$$

$$\boldsymbol{\Omega}\left(h_U\left(x\right)\right) = \mathbf{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \mathbf{V}\left(h_U\left(x\right),i\right) + \mathbf{V}\left(h_U\left(x\right),i\right)$$

$$\mathbf{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \mathbf{V}\left(h_U\left(x\right),i\right) = \boldsymbol{\Omega}\left(h_U\left(x\right)\right) - \mathbf{V}\left(h_U\left(x\right),i\right)$$

Using this in the term in the square brackets

$$UE \text{ Hires} = \lambda^F v\left(x\right)\phi\left[\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - \mathbf{V}\left(h_U\left(x\right), i\right)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}$$

Under **C1**, the value going to the hired worker is $\mathbf{V}\left(h_U\left(x\right), i\right) = U$, using this:

$$UE \text{ Hires} = \lambda^F v\left(x\right)\phi\left[\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - U\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}$$

(v) UE **Threats**

Recall that

$$UE \text{ Threats} \quad = \quad \lambda^F v\left(x\right)\phi\left[\mathbf{J}\left(t_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \mathbf{V}\left(t_U\left(x\right), i\right) - \Omega\left(x\right)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}}$$

Using the definition of $\mathbf{\Omega}(t_U(x))$, we can re-write this term as

$$UE \text{ Threats} \quad = \quad \lambda^F v\left(x\right)\phi\left[\mathbf{\Omega}\left(t_U\left(x\right)\right) - \Omega(x)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}}$$

Now using our result in condition **C1** that $\mathbf{\Omega}\left(t_U\left(x\right)\right) = \Omega(x)$, we can conclude that

$$UE \text{ Threats} = 0$$

(vi) EE **Hires**

$$EE \text{ Hires} = \lambda^F v\left(x\right)\left(1-\phi\right)\int_{x \in \mathcal{Q}^E(x',i')} \left[\mathbf{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \mathbf{V}\left(h_E\left(x',i',x\right), i\right) - \Omega\left(x\right)\right] dG\left(x',i'\right)$$

Now by definition

$$\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) = \mathbf{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(h_E(x',i',x))} \mathbf{V}\left(h_E\left(x',i',x\right), i\right)$$

$$\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) = \left[\mathbf{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \mathbf{V}\left(h_E\left(x',i',x\right), i\right)\right] + \mathbf{V}\left(h_E\left(x',i',x\right), i\right)$$

$$\mathbf{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \mathbf{V}\left(h_E\left(x',i',x\right), i\right) = \mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \mathbf{V}\left(h_E\left(x',i',x\right), i\right)$$

Using this in the term in the square brackets

$$EE \text{ Hires} = \lambda^F v\left(x\right)\left(1-\phi\right)\int_{x \in \mathcal{Q}^E(x',i')} \left[\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right) - \mathbf{V}\left(h_E\left(x',i',x\right), i\right)\right] dG\left(x',i'\right)$$

Under **C2**, the value going to the hired worker is $\mathbf{V}\left(h_E\left(x',i',x\right),i\right) = \Omega\left(x'\right) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)$, using this:

$$EE\text{ Hires} = \lambda^F v\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^E\left(x',i'\right)}\left[\left[\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega\left(x\right)\right]-\left[\Omega\left(x'\right)-\mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)\right]\right]dG\left(x',i'\right)$$

(vii) EE **Threats**

Recall that

$$EE\text{ Threats} \;=\; \lambda^F v\left(x\right)\left(1-\phi\right)\int_{x\notin\mathcal{Q}^E\left(x',i'\right)}\left[\mathbf{J}\left(t_E\left(x',i',x\right)\right)+\sum_{i=1}^{n(x)}\mathbf{V}\left(t_E\left(x',i',x\right),i\right)-J\left(x\right)-\bar{V}\left(x\right)\right]dG\left(x',i'\right)$$

Using the definition of $\mathbf{\Omega}(t_E(x',i',x))$, we obtain

$$EE\text{ Threats} \;=\; \lambda^F v\left(x\right)\left(1-\phi\right)\int_{x\notin\mathcal{Q}^E\left(x',i'\right)}\left[\mathbf{\Omega}\left(t_E\left(x',i',x\right)\right)-\Omega(x)\right]dG\left(x',i'\right)$$

Now using the result in condition C1 that $\mathbf{\Omega}\left(t_E\left(x',i',x\right)\right)=\Omega(x)$, we obtain that

$$EE\text{ Threats} = 0$$

(viii) Productivity shocks

$$\text{Shocks} = \mathbb{E}_\eta\left[\mathbf{J}\left(g_z\left(x,\eta\right)\right)+\sum_{i=1}^{n(x)}\mathbf{V}\left(g_z\left(x,\eta\right),i\right)\right]-\Omega\left(x\right)$$

where the term in square bracket is:

$$\mathbf{J}\left(g_z\left(x,\eta\right)\right)+\sum_{i=1}^{n(x)}\mathbf{V}\left(g_z\left(x,\eta\right),i\right)=\mathbf{\Omega}\left(g_z\left(x,\eta\right)\right)$$

which gives

$$\text{Shocks} = \mathbb{E}_\eta\left[\mathbf{\Omega}\left(g_z\left(x,\eta\right)\right)-\Omega\left(x\right)\right]$$

Final Bellman equation, sets, vacancy policy in x

Under our assumption **A-TE**, a full characterization of the Bellman equation is then, for $x \in \bar{\mathcal{E}}$ the complement of the exit set, given by:

$$
\begin{array}{rcl}
\rho\Omega\left(x\right) & = & y\left(z(x), n\left(x\right)\right) - c\left(v\left(x\right), n\left(x\right), z(x)\right) \\
\text{Destructions} & & -\delta \displaystyle\sum_{i=1}^{n(x)} \left[\Omega\left(x\right) - \mathbf{\Omega}\left(d(x,i)\right) - U\right] \\
UE \text{ Hires} & & +\lambda^F v\left(x\right)\phi\left[\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - U\right]\cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
EE \text{ Hires} & & +\lambda^F v\left(x\right)\left(1-\phi\right)\displaystyle\int_{x\in\mathcal{Q}^E(x',i')} \left[\left[\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right)\right] - \left[\Omega\left(x'\right) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)\right]\right] dG\left(x',i'\right) \\
\text{Shocks} & & +\mathbb{E}_\eta\left[\mathbf{\Omega}\left(g_z\left(x,\eta\right)\right) - \Omega\left(x\right)\right]
\end{array}
$$

with the sets

$$
\begin{aligned}
\mathcal{Q}^U &= \left\{(x,i)\middle|\Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=1\right]\right)\right),i\right) + U > \Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right)\right.\right.\right.\right.\right. \\
\mathcal{L} &= \left\{(x,i)\middle|\Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=1\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),i\right) + U > \Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=0\right]\right)\circ\left(1-q\right.\right.\right.\right. \\
\mathcal{E} &= \left\{x\middle|\vartheta + n\left(s(x,\kappa(x))\right)\cdot U \geq \Omega(s(x,\kappa(x)))\right\} \\
\mathcal{A} &= \left\{x\middle|\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) \geq U\right\} \\
\mathcal{Q}^E\left(x',i'\right) &= \left\{x\middle|\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right) \geq \Omega\left(x'\right) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)\right\}
\end{aligned}
$$

and—as per **(C6)**—the vacancy policy $v\left(x\right)$ is given by the solution to the following:

$$
\begin{array}{rcl}
\dfrac{c_v\left(v\left(x\right), n\left(x\right)\right)}{\lambda^F} & = & \phi\left[\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right)\right]\cdot\mathbb{I}_{\{x\in\mathcal{A}\}} \\
& + & \left(1-\phi\right)\displaystyle\int_{x\in\mathcal{Q}^E(x',i')}\left[\left[\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right)\right] - \left[\Omega\left(x'\right) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)\right]\right] dG\left(x',i'\right)
\end{array}
$$

In continuous time, the exit decision is captured by $x \in \mathcal{E}$. The Bellman equation above holds exactly for $x \in \mathcal{E}^c$. Exit is accounted for in the "bold" continuation values, which all include the possible exit decision should the firm's state fall into $\mathcal{E}$ after an event.

A.4 Reducing the state-vector

We proceed in three steps. First, we isolate (n,z) in the state vector x by writing $x = (n, z, \chi)$ where χ is all other terms in x. Second, we introduce functions that update χ following events to the firm and worker. Third, we argue that χ is a redundant state. This delivers the final Bellman equation for the joint value function for the discrete work-force model, equation ($A1$).

A.4.1 Isolate (n,z) in the state vector

It is immediate that x should contain at least the pair (n,z). Call everything else χ. Then we express $x = (n,z,\chi)$. Making this substitution into the above conditions:

$$
\begin{aligned}
\rho\Omega(n,z,\chi) \quad &= \quad y(n,z) - c(v(n,z,\chi),n) \\
\text{Destructions} \quad & \quad -\delta \sum_{i=1}^{n(x)} \left[\Omega(n,z,\chi) - \mathbf{\Omega}(d(n,z,\chi,i)) - U\right] \\
UE \text{ Hires} \quad & \quad +\lambda^F v(n,z,\chi)\,\phi\left[\mathbf{\Omega}(h_U(n,z,\chi)) - \Omega(n,z,\chi) - U\right]\cdot \mathbb{I}_{\{(n,z,\chi)\in\mathcal{A}\}} \\
EE \text{ Hires} \quad & \quad +\lambda^F v(n,z,\chi)(1-\phi)\int_{(n,z,\chi)\in\mathcal{Q}(n',z',\chi',i')} \Big[\left[\mathbf{\Omega}\left(h_E(n',z',\chi',i',n,z,\chi)\right) - \Omega(n,z,\chi)\right] \\
& \qquad\qquad - \left[\Omega(n',z',\chi',i') - \mathbf{\Omega}\left(q_E(n',z',\chi',i',n,z,\chi)\right)\right]\Big] \\
& \qquad\qquad \cdot dG(n',z',\chi',i') \\
\text{Shocks} \quad & \quad +\mathbb{E}_\eta\left[\mathbf{\Omega}(g_z(n,z,\chi,\eta)) - \Omega(n,z,\chi)\right]
\end{aligned}
$$

with sets

$$
\begin{aligned}
\mathcal{Q}^U &= \Big\{(n,z,\chi,i)\Big|\Omega\left(s\left(n,z,\chi,(1-\ell(n,z,\chi))\circ(1-[q_{U,-i}(n,z,\chi);q_{U,i}(n,z,\chi)=1])\right),i\right)+U \\
&\qquad > \Omega\left(s\left(n,z,\chi,(1-\ell(n,z,\chi))\circ(1-[q_{U,-i}(n,z,\chi);q_{U,i}(n,z,\chi)=0])\right),i\right)\Big\} \\
\mathcal{L} &= \Big\{(n,z,\chi,i)\Big|\Omega\left(s\left(n,z,\chi,(1-[\ell(n,z,\chi);\ell_i(n,z,\chi)=1])\circ(1-q_U(n,z,\chi))\right),i\right)+U \\
&\qquad > \Omega\left(s\left(n,z,\chi,(1-[\ell(n,z,\chi);\ell_i(n,z,\chi)=0])\circ(1-q_U(n,z,\chi))\right),i\right)\Big\} \\
\mathcal{E} &= \left\{n,z,\chi\middle|\vartheta + n\left(s(n,z,\chi,\kappa(n,z,\chi))\right)\cdot U \geq \Omega(s(n,z,\chi,\kappa(n,z,\chi)))\right\} \\
\mathcal{A} &= \left\{n,z,\chi\middle|\Omega(h_U(n,z,\chi)) - \Omega(n,z,\chi) \geq U\right\} \\
\mathcal{Q}^E(n',z',\chi',i') &= \left\{n,z,\chi\middle|\mathbf{\Omega}\left(h_E(n',z',\chi',i',n,z,\chi)\right) - \Omega(n,z,\chi) \geq \Omega(n',z',\chi',i') - \mathbf{\Omega}\left(q_E(n',z',\chi',i',n,z,\chi)\right)\right\}
\end{aligned}
$$

and vacancy posting

$$\begin{aligned}\frac{c_v\left(v\left(n,z,\chi\right),n,z\right)}{\lambda^F} &= \phi\left[\boldsymbol{\Omega}\left(h_U\left(n,z,\chi\right)\right)-\Omega\left(n,z,\chi\right)\right]\cdot\mathbb{I}_{\{(n,z,\chi)\in\mathcal{A}\}}\\ &+(1-\phi)\int_{(n,z,\chi)\in\mathcal{Q}(n',z',\chi',i')}\Big[\left[\boldsymbol{\Omega}\left(h_E\left(n',z',\chi',i',n,z,\chi\right)\right)-\Omega\left(n,z,\chi\right)\right]\\ &\qquad -\left[\Omega\left(n',z',\chi',i'\right)-\boldsymbol{\Omega}\left(q_E\left(n',z',\chi',i',n,z,\chi\right)\right)\right]\Big]\\ &\qquad\cdot dG\left(n',z',\chi',i'\right)\end{aligned}$$

A.4.2 Introduce functions that update the residual χ

We define the following functions given that we know how n changes in each of the cases

$$\begin{aligned}s(n,z,\chi,\kappa(n,z,\chi)) &= (\mathcal{N}(n,z,\chi),z,s^{\chi}(n,z,\chi))\\ d\left(n,z,\chi,i\right) &= \left(n-1,z,d^{\chi}\left(n,z,\chi,i\right)\right)\\ s\left(n,z,\chi,\left(1-\ell\left(n,z,\chi\right)\right)\circ\left(1-\left[q_{U,-i}\left(n,z,\chi\right);q_{U,i}\left(n,z,\chi\right)=1\right]\right)\right) &= (\mathcal{N}(n,z,\chi)-\tau_1(n,z,\chi),z,\tau_1^{\chi}(n,z,\chi,i))\\ s\left(n,z,\chi,\left(1-\ell\left(n,z,\chi\right)\right)\circ\left(1-\left[q_{U,-i}\left(n,z,\chi\right);q_{U,i}\left(n,z,\chi\right)=0\right]\right)\right) &= (\mathcal{N}(n,z,\chi),z,\tau_0^{\chi}(n,z,\chi,i))\\ s\left(n,z,\chi,\left(1-\left[\ell\left(n,z,\chi\right);\ell_i\left(n,z,\chi\right)=1\right]\right)\circ\left(1-q_U\left(n,z,\chi\right)\right)\right) &= (\mathcal{N}(n,z,\chi)-\eta_1(n,z,\chi),z,\eta_1^{\chi}(n,z,\chi,i))\\ s\left(n,z,\chi,\left(1-\left[\ell\left(n,z,\chi\right);\ell_i\left(n,z,\chi\right)=0\right]\right)\circ\left(1-q_U\left(n,z,\chi\right)\right)\right) &= (\mathcal{N}(n,z,\chi)(n,z,\chi),z,\eta_0^{\chi}(n,z,\chi,i))\\ h_U\left(n,z,\chi\right) &= \left(n+1,z,h_U^{\chi}\left(n,z,\chi\right)\right)\\ h_E\left(z',\chi',i',n,z,\chi,n'\right) &= \left(n+1,z,h_E^{\chi}\left(n',z',\chi',i',n,z,\chi\right)\right)\\ q_E\left(n',z',\chi',i',n,z,\chi\right) &= \left(n'-1,z',q_E^{\chi}\left(n',z',\chi',i',n,z,\chi\right)\right)\\ G\left(n',z',\chi',i'\right) &= \frac{1}{n'}G\left(z',n',\chi'\right)\\ g_z\left(n,z,\chi,\eta\right) &= \Big(n,g_z\left(z,\eta\right),g_z^{\chi}\left(n,z,\chi,\eta\right)\Big)\end{aligned}$$

In particular, $\mathcal{N}(n,z,\chi)$ is the number of workers the firm retains after endogenous quits and layoffs. It solves

$$\mathcal{N}(n,z)=\arg\max_{k\in\{0,\ldots,n\}}\Omega(k,z,\chi)+(n-k)U$$

$\tau_1(n,z,\chi),\eta_1(n,z,\chi)\in\{0,1\}$. $\tau_1(n,z,\chi)=0$ if $\ell_i(n,z,\chi)=1$. Similarly, $\eta_1(n,z,\chi)=0$ if $q_{U,i}(n,z,\chi)=1$.

Using these definitions in the Bellman equation above:

$$
\begin{aligned}
\rho\Omega\left(n,z,\chi\right) \quad &= \quad y\left(n,z\right)-c\left(v\left(n,z,\chi\right),n,z\right) \\
\text{Destructions} \quad & \quad -\delta\sum_{i=1}^{n(x)}\left[\Omega\left(n,z,\chi\right)-\mathbf{\Omega}\left(n-1,z,s^{\chi}\left(n,z,\chi,i\right)\right)-U\right] \\
UE\text{ Hires} \quad & \quad +\lambda^{F}v\left(n,z,\chi\right)\phi\left[\Omega\left(n+1,z,h_{U}^{\chi}\left(n,z,\chi\right)\right)-\Omega\left(n,z,\chi\right)-U\right]\cdot\mathbb{I}_{\{(n,z,\chi)\in\mathcal{A}\}} \\
EE\text{ Hires} \quad & \quad +\lambda^{F}v\left(n,z,\chi\right)\left(1-\phi\right)\int_{(n,z,\chi)\in\mathcal{Q}(n',z',\chi',i')}\Big[\left[\mathbf{\Omega}\left(n+1,z,h_{E}^{\chi}\left(n',z',\chi',i',n,z,\chi\right)\right)-\Omega\left(n,z,\chi\right)\right] \\
& \qquad\qquad -\left[\Omega\left(n',z',\chi',i'\right)-\mathbf{\Omega}\left(n'-1,z',q_{E}^{\chi}\left(n',z',\chi',i',n,z,\chi\right)\right)\right]\Big] \\
& \qquad\qquad \cdot dG\left(n',z',\chi',i'\right) \\
\text{Shocks} \quad & \quad +\mathbb{E}_{\eta}\left[\mathbf{\Omega}\left(n,g_{z}\left(z,\eta\right),g_{z}^{\chi}\left(n,z,\chi,\eta\right)\right)-\Omega\left(n,z,\chi\right)\right]
\end{aligned}
$$

and sets

$$
\begin{aligned}
\mathcal{E} &= \left\{n,z,\chi \,\middle|\, \vartheta+\mathcal{N}(n,z,\chi)\cdot U\geq\Omega(\mathcal{N}(n,z,\chi),z,s^{\chi}(n,z,\chi))\right\} \\
\mathcal{Q}^{U} &= \Big\{(n,z,\chi,i)\,\Big|\,\Omega(\mathcal{N}(n,z,\chi)-\tau_{1}(n,z,\chi),z,\tau_{1}^{\chi}(n,z,\chi,i))+U \\
&\qquad\qquad > \Omega(\mathcal{N}(n,z,\chi),z,\tau_{0}^{\chi}(n,z,\chi,i))\Big\} \\
\mathcal{L} &= \Big\{(n,z,\chi,i)\,\Big|\,\Omega(\mathcal{N}(n,z,\chi)-\eta_{1}(n,z,\chi),z,\eta_{1}^{\chi}(n,z,\chi,i))+U \\
&\qquad\qquad > \Omega(\mathcal{N}(n,z,\chi),z,\eta_{0}^{\chi}(n,z,\chi,i))\Big\} \\
\mathcal{A} &= \left\{n,z,\chi\,\middle|\,\Omega\left(n+1,z,h_{U}^{\chi}\left(n,z,\chi\right)\right)-\Omega\left(n,z,\chi\right)\geq U\right\} \\
\mathcal{Q}^{E}\left(n',z',\chi',i'\right) &= \Big\{n,z,\chi\,\Big|\,\mathbf{\Omega}\left(n+1,z,h_{E}^{\chi}\left(n,z,\chi,n',z',\chi',i'\right)\right)-\Omega\left(n,z,\chi\right) \\
&\qquad\qquad \geq\Omega\left(n',z',\chi',i'\right)-\Omega\left(n'-1,z',p^{\chi}\left(n',z',\chi',i',n,z,\chi\right)\right)\Big\}
\end{aligned}
$$

and the definition

$$
\mathcal{N}(n,z,\chi)=\arg\max_{k\in\{0,\dots,n\}}\Omega(k,z,\chi)+(n-k)U
$$

and vacancy posting

$$
\begin{aligned}
\frac{c_v\left(v\left(n,z,\chi\right),n,z\right)}{\lambda^F} &= \phi\left[\boldsymbol{\Omega}\left(n+1,z,h_U^{\chi}\left(n,z,\chi\right)\right)-\Omega\left(n,z,\chi\right)\right]\cdot\mathbb{I}_{\{(n,z,\chi)\in\mathcal{A}\}} \\
&+\left(1-\phi\right)\int_{(n,z,\chi)\in\mathcal{Q}(n',z',\chi',i')}\Big[\left[\boldsymbol{\Omega}\left(n+1,z,h_E^{\chi}\left(n,z,\chi,n',z',\chi',i'\right)\right)-\Omega\left(n,z,\chi\right)\right] \\
&\qquad -\left[\Omega\left(n',z',\chi',i'\right)-\boldsymbol{\Omega}\left(n'-1,z',q_E^{\chi}\left(n',z',\chi',i',n,z,\chi\right)\right)\right]\Big] \\
&\qquad \cdot\, dG\left(n',z',\chi',i'\right)
\end{aligned}
$$

A.4.3 Argue that (χ, i) are a redundant state

The system above defines a functional fixed point equation. Inspection of the Bellman equation reveals that χ has no *direct* impact on the flow payoff, continuation values, or mobility sets. Its only impact is through the dependence of Ω on χ. This observation implies that χ is a redundant state, and can be removed from the fixed point equation

The same argument ensures that the worker index i is redundat as well.

A.4.4 Bellman equation without (χ, i)

We can re-write our Bellman equation for $(n,z)\in\mathcal{E}^c$ as:

$$
\begin{aligned}
\rho\Omega\left(n,z\right) &= y\left(n,z\right)-c\left(v\left(n,z\right),n\right) \\
\text{Destructions}\quad & -\delta\sum_{i=1}^{n}\left[\Omega\left(n,z\right)-\boldsymbol{\Omega}\left(n-1,z\right)-U\right] \\
\text{Retentions}\quad & +\lambda^E\sum_{i=1}^{n}\int_{(n',z')\in\mathcal{R}(n,z)}\left[\boldsymbol{\Omega}\left(n,z\right)-\Omega\left(n,z\right)\right]dF\left(x'\right) \\
UE\text{ Hires}\quad & +\lambda^F v\left(n,z\right)\phi\left[\boldsymbol{\Omega}\left(n+1,z\right)-\Omega\left(n,z\right)-U\right]\cdot\mathbb{I}_{\{(n,z)\in\mathcal{A}\}} \\
EE\text{ Hires}\quad & +\lambda^F v\left(n,z\right)\left(1-\phi\right)\int_{(n,z)\in\mathcal{Q}(n',z')}\Bigg[\left[\boldsymbol{\Omega}\left(n+1,z\right)-\Omega\left(n,z\right)\right]-\left[\Omega\left(n',z'\right)-\boldsymbol{\Omega}\left(n'-1,z'\right)\right]\Bigg]d\tilde{G}\left(n',z\right. \\
\text{Shocks}\quad & +\mathbb{E}_{\eta}\left[\boldsymbol{\Omega}\left(n,g_z\left(z,\eta\right)\right)-\Omega\left(n,z\right)\right]
\end{aligned}
$$

with the sets

$$\mathcal{E}^c = \left\{ n,z \middle| \Omega(\mathcal{N}(n,z)) \geq \vartheta + \mathcal{N}(n,z)U \right\}$$

$$\mathcal{L} = \mathcal{Q}^U = \left\{ n,z \middle| \Omega(\mathcal{N}(n,z),z) - \mathbf{\Omega}(\mathcal{N}(n,z)-1,z) > U \right\}$$

$$\mathcal{A} = \left\{ n,z \middle| \mathbf{\Omega}(n+1,z) - \Omega(n,z) \geq U \right\}$$

$$\mathcal{Q}^E(n',z') = \left\{ n,z \middle| \mathbf{\Omega}(n+1,z) - \Omega(n,z) \geq \Omega(n',z') - \mathbf{\Omega}(n'-1,z') \right\}$$

and the definition

$$\mathcal{N}(n,z) = \arg \max_{k \in \{0,\ldots,n\}} \Omega(k,z) + (n-k)U$$

and the vacancy policy function:

$$\begin{aligned}\frac{c_v(v(n,z),n,z)}{\lambda^F} &= \phi\left[\mathbf{\Omega}(n+1,z) - \Omega(n,z)\right] \cdot \mathbb{I}_{\{(n,z)\in\mathcal{A}\}} \\ &+ (1-\phi)\int_{(n,z)\in\mathcal{Q}(n',z')} \Big[\left[\mathbf{\Omega}(n+1,z) - \Omega(n,z)\right] - \left[\Omega(n',z') - \mathbf{\Omega}(n'-1,z')\right] \Big] d\tilde{G}(n',z')\end{aligned}$$

A.4.5 Expressing "bold" values

In this step we express "bold" values as simple functions of non-bold values.

- From the definition of the exit and quit sets $\mathcal{E}$, $\mathcal{Q}^U$, we can express:

$$\mathbf{\Omega}(n,z) = \max\left\{ \underbrace{\Omega(n,z)}_{\text{Operate}}, \underbrace{\mathbf{\Omega}(n-1,z)+U}_{\text{Separate one worker and re-evaluate}}, \underbrace{\vartheta + nU}_{\text{Exit}} \right\}$$

- We can iterate on this equation. To see the logic, consider the first few steps.

$$\begin{aligned}\mathbf{\Omega}(n,z) &= \max\left\{ \Omega(n,z)\, ,\, \mathbf{\Omega}(n-1,z)+U\, ,\, \vartheta+nU \right\} \\ &= \max\left\{ \Omega(n,z)\, ,\, \max\left\{ \Omega(n-1,z)\, ,\, \mathbf{\Omega}(n-2,z)+U\, ,\, \vartheta+(n-1)U \right\} + U\, ,\, \vartheta+nU \right\} \\ &= \max\left\{ \Omega(n,z)\, ,\Omega(n-1,z)+U\, ,\, \mathbf{\Omega}(n-2,z)+2U\, ,\, \vartheta+(n-1)U+U\, ,\, \vartheta+nU \right\} \\ &= \max\left\{ \Omega(n,z)\, ,\Omega(n-1,z)+U\, ,\, \mathbf{\Omega}(n-2,z)+2U\, ,\, \vartheta+nU \right\}\end{aligned}$$

- By recursion, it is easy to see that

$$\begin{aligned}\mathbf{\Omega}(n,z) &= \max\left\{\Omega(\mathcal{N}(n,z),z)+(n-\mathcal{N}(n,z))\cdot U\ ,\ \vartheta+nU\right\}\\ &= \max\left\{\max_{k\in\{0,\ldots,n\}}\Omega(k,z)+(n-k)U\ ,\ \vartheta+nU\right\}\end{aligned}$$

where recall that

$$\mathcal{N}(n,z)=\arg\max_{k\in\{0,\ldots,n\}}\Omega(k,z)+(n-k)U$$

A.4.6 Final Bellman equation of the discrete work-force model

Finally, we can express our Bellman equation as a joint system of equations:

$$\begin{aligned}\rho\Omega\left(n,z\right)=\max_{v\geq 0}\quad & y\left(n,z\right)-c\left(v,n,z\right) \qquad \text{(A1)}\\ \text{Destructions}\quad & -\delta n\left[\Omega\left(n,z\right)-\mathbf{\Omega}\left(n-1,z\right)-U\right]\\ UE\text{ Hires}\quad & +\lambda^{F}v\phi\left[\mathbf{\Omega}\left(n+1,z\right)-\Omega\left(n,z\right)-U\right]\cdot\mathbb{I}_{\{(n,z)\in\mathcal{A}\}}\\ EE\text{ Hires}\quad & +\lambda^{F}v\left(1-\phi\right)\int_{(n,z)\in\mathcal{Q}(n',z')}\Big[\left[\mathbf{\Omega}\left(n+1,z\right)-\Omega\left(n,z\right)\right]-\left[\Omega\left(n',z'\right)-\mathbf{\Omega}\left(n'-1,z'\right)\right]\Big]d\tilde{G}\left(n',z'\right)\\ \text{Shocks}\quad & +\mathbb{E}_{\eta}\left[\mathbf{\Omega}\left(n,g_{z}\left(z,\eta\right)\right)-\Omega\left(n,z\right)\right]\end{aligned}$$

with the sets

$$\begin{aligned}\mathcal{E} &= \left\{n,z\,\middle|\,\max_{k\in\{0,\ldots,n\}}\Omega(k,z)+(n-k)U<\vartheta+nU\right\}\\ \mathcal{A} &= \left\{n,z\,\middle|\,\mathbf{\Omega}\left(n+1,z\right)-\Omega\left(n,z\right)\geq U\right\}\\ \mathcal{Q}^{U} &= \left\{n,z\,\middle|\,\Omega\left(n,z\right)-\mathbf{\Omega}\left(n-1,z\right)>U\right\}\\ \mathcal{Q}^{E}\left(n',z'\right) &= \left\{n,z\,\middle|\,\mathbf{\Omega}\left(n+1,z\right)-\Omega\left(n,z\right)\geq\Omega\left(n',z'\right)-\mathbf{\Omega}\left(n'-1,z'\right)\right\}\end{aligned}$$

and the definition:

$$\mathbf{\Omega}(n,z) = \max\left\{\max_{k\in\{0,\ldots,n\}}\Omega(k,z)+(n-k)U\ ,\ \vartheta+nU\right\}$$

A.5 Continuous work-force limit

To do so, we proceed in three steps:

(A.5.1) Define worker size and the renormalization

(A.5.2) Take the limit as worker size goes to zero

(A.5.3) Introduce a continuous productivity process.

A.5.1 Define worker size and the renormalization

We denote the "size" of a worker by Δ. We make the following renormalization: we define

$$\begin{aligned}
\omega(m,z) &= \Omega\left(\frac{m}{\Delta},z\right) \\
\mathcal{Y}(m,z) &= y\left(\frac{m}{\Delta},z\right) \\
\mathcal{C}(m,z) &= c\left(\frac{v}{\Delta},\frac{m}{\Delta},z\right)
\end{aligned}$$

These definition imply

$$\begin{aligned}
\Omega(n,z) &= \omega(n\Delta,z) \\
y(n,z) &= \mathcal{Y}(n\Delta,z) \\
c(v,n,z) &= \mathcal{C}(v\Delta,n\Delta,z)
\end{aligned}$$

In addition, the value of unemployment solves

$$\rho U = b$$

Define

$$\mathcal{U} = \frac{b}{\rho\Delta} = \frac{U}{\Delta}$$

and

$$\theta = \frac{\vartheta}{\Delta}$$

Substituting these definitions into the Bellman equation, we obtain

$$\begin{aligned}
\rho\omega\left(n\Delta,z\right) = \max_{v\Delta\geq 0} \quad & \mathcal{Y}\left(n\Delta,z\right) - \mathcal{C}\left(v\Delta,n\Delta,z\right) \\
\text{Destructions} \quad & -\delta n\Delta\left[\frac{\omega\left(n\Delta,z\right)-\omega\left(n\Delta-\Delta,z\right)}{\Delta} - \mathcal{U}\right] \\
UE\text{ Hires} \quad & +\lambda^{F} v\Delta\phi\left[\frac{\omega\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta} - \mathcal{U}\right]\cdot \mathbb{I}_{\{(n\Delta,z)\in\mathcal{A}\}} \\
EE\text{ Hires} \quad & +\lambda^{F} v\Delta\left(1-\phi\right)\int_{(n\Delta,z)\in\mathcal{Q}(n'\Delta,z')}\left[\frac{\omega\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta} - \frac{\omega\left(n'\Delta,z'\right)-\omega\left(n'\Delta-1\Delta,z'\right)}{\Delta}\right]d\tilde{G} \\
\text{Shocks} \quad & +\mathbb{E}_{\eta}\left[\omega\left(n\Delta,g_{z}\left(z,\eta\right)\right)-\omega\left(n\Delta,z\right)\right]
\end{aligned}$$

with the set definitions

$$\begin{aligned}
\mathcal{E} &= \left\{ n\Delta,z \,\middle|\, \max_{k\Delta\in\{0,\ldots,n\Delta\}} \omega(k\Delta,z) + (n\Delta-k\Delta)\mathcal{U} < \theta + n\Delta\mathcal{U} \right\} \\
\mathcal{A} &= \left\{ n\Delta,z \,\middle|\, \frac{\omega\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta} \geq \mathcal{U} \right\} \\
\mathcal{Q}^{U} &= \left\{ n\Delta,z \,\middle|\, \frac{\omega\left(n\Delta,z\right)-\omega\left(n\Delta-\Delta,z\right)}{\Delta} > \mathcal{U} \right\} \\
\mathcal{Q}^{E}\left(n'\Delta,z'\right) &= \left\{ n\Delta,z \,\middle|\, \frac{\omega\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta} \geq \frac{\omega\left(n'\Delta,z'\right)-\omega\left(n'\Delta-\Delta,z'\right)}{\Delta} \right\}
\end{aligned}$$

and the definition:

$$\omega(n\Delta,z) \quad = \quad \max\left\{ \max_{k\Delta\in\{0,\ldots,n\Delta\}} \omega(k\Delta,z) + (n\Delta-k\Delta)\mathcal{U}\ ,\ \theta + n\Delta\mathcal{U} \right\}$$

A.5.2 Continuous limit as worker size goes to zero

Now we take the limit $\Delta \to 0$, holding $m = n\Delta$ fixed. We note $\hat{v} = \lim_{\Delta\to 0} v\Delta$. We see derivatives appear. We denote $\omega_m(m,z) = \frac{\partial\omega}{\partial m}(m,z)$.

First, we note that the following limit obtains:

$$\omega(m,z) \quad := \quad \max\left\{ \max_{k\in[0,m]} \omega(k,z) + (n-k)\mathcal{U}\ ,\ \theta + n\Delta\mathcal{U} \right\}$$

In particular, the exit set limits to

$$\mathcal{E} = \left\{ m, z \left| \max_{k\in[0,m]} \omega(k,z) + (n-k)\mathcal{U} < \theta + m\mathcal{U} \right. \right\}$$

In equilibrium, the $\omega(m,z)$ on the right-hand-side of the Bellman equation is the result of an endogenous quit/layoff (recall that we write the Bellman equation after the endogenous quit/layoff/exit decisions). This observation implies, that for any $(m,z) \in \bar{\mathcal{E}}^{\circ}$, the *interior* of the continuation set, there is always $\bar{\Delta} > 0$: such that for any $\Delta \leq \bar{\Delta}$:

$$\omega(m \pm \Delta, z) = \omega(m \pm \Delta, z)$$

Using this observation in the Bellman equation, we can obtain derivatives on the right-hand-side. We obtain

$$\begin{aligned}
\rho\omega\left(m,z\right) = \max_{\hat{v}\geq 0} \quad & \mathcal{Y}\left(m,z\right) - \mathcal{C}\left(\hat{v},m,z\right) \\
\text{Destructions} \quad & -\delta m[\omega_m(m,z) - \mathcal{U}] \\
UE \text{ Hires} \quad & +\lambda^F \hat{v}\phi\left[\omega_m(m,z) - \mathcal{U}\right] \cdot \mathbb{I}_{\{(m,z)\in\mathcal{A}\}} \\
EE \text{ Hires} \quad & +\lambda^F \hat{v}\left(1-\phi\right) \int_{(m,z)\in\mathcal{Q}^E(m',z')} \left[\omega_m(m,z) - \omega_m(m',z')\right] d\tilde{G}\left(m',z'\right) \\
\text{Shocks} \quad & +\mathbb{E}_\eta\left[\omega\left(m, g_z\left(z,\eta\right)\right) - \omega\left(m,z\right)\right]
\end{aligned}$$

with the set definitions

$$\begin{aligned}
\mathcal{E} &= \left\{ m, z \left| \max_{k\in[0,m]} \omega(k,z) + (n-k)\mathcal{U} < \theta + m\mathcal{U} \right. \right\} \\
\mathcal{A} &= \left\{ m, z \left| \omega_m(m,z) \geq \mathcal{U} \right. \right\} \\
\mathcal{Q}^U &= \left\{ m, z \left| \omega_m(m,z) > \mathcal{U} \right. \right\} = \bar{A} \text{ , the complement of } \mathcal{E} \\
\mathcal{Q}^E\left(m,z'\right) &= \left\{ m, z \left| \omega_m(m,z) - \omega_m(m',z') \right. \right\}
\end{aligned}$$

and the definition:

$$\omega(m,z) \quad = \quad \max\left\{ \max_{k\in[0,m]} \omega(k,z) + (m-k)\mathcal{U} \text{ , } \theta + m\mathcal{U} \right\}$$

A.5.3 Continuous productivity process

We now specialize to a continuous productivity process, as this makes the formulation of the problem very economical. It allows to simplify the contribution of productivity shocks and get rid of the "bold" notation. We suppose that productivity follows a diffusion process:

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$$

In this case, for any (m, z) in the interior of the continuation set, productivity shocks in the interval $[t, t + dt]$ cannot not move the firm towards a region in which it would endogenously separate or exit, when dt is small enough. In this case, we can write economically:

$$\begin{aligned}
\rho\omega\,(m,z) = \max_{v\geq 0} \quad & \mathcal{Y}\,(m,z) - \mathcal{C}\,(v,m,z) \\
\text{Destructions} \quad & -\delta m[\omega_m(m,z) - \mathcal{U}] \\
UE \text{ Hires} \quad & +\lambda^F v\phi\,[\omega_m(m,z) - \mathcal{U}] \\
EE \text{ Hires} \quad & +\lambda^F v\,(1-\phi) \int \max\left[\omega_m(m,z) - \omega_m(m',z')\;,\;0\right] d\tilde{G}\,(m',z') \\
\text{Shocks} \quad & +\mu(z)\omega_z(m,z) + \frac{\sigma(z)^2}{2}\omega_{zz}(m,z) \\
\text{s.t.} \quad & \\
\text{No Exit} \quad & \omega(m,z) \geq \theta + m\mathcal{U} \\
\text{No Separations} \quad & \omega_m(m,z) \geq \mathcal{U}
\end{aligned}$$

To make the notation more comparable, we slightly abuse notation and use the same letters as before, but now for the continuous workforce case. We obtain finally:

$$
\begin{aligned}
\rho\Omega\left(n,z\right) = \max_{v\geq 0} \quad & y\left(n,z\right) - c\left(v,n,z\right) \\
\text{Destructions} \quad & -\delta n[\Omega_n(n,z) - U] \\
UE \text{ Hires} \quad & +\lambda^F v\phi\left[\Omega_n(n,z) - U\right] \\
EE \text{ Hires} \quad & +\lambda^F v\left(1-\phi\right)\int \max\left[\Omega_n(n,z) - \Omega_n(n',z')\ ,\ 0\right] d\tilde{G}\left(n',z'\right) \\
\text{Shocks} \quad & +\mu(z)\Omega_z(m,z) + \frac{\sigma(z)^2}{2}\Omega_{zz}(m,z) \\
& \text{s.t.} \\
\text{No Exit} \quad & \Omega(n,z) \geq \vartheta + nU \\
\text{No Separations} \quad & \Omega_n(n,z) \geq U
\end{aligned}
$$

When the coalition hits $\Omega_n(n,z) = U$, it endogenous separates worker to stay on that frontier. It exits when it hits the frontier $\Omega(n,z) = \vartheta + nU$.

B Additional mathematical details

To be completed

C Additional figures and tables

To be completed

D Algorithm

To effectively reduce the dimension of the parameter space by one, we slightly recast the problem to view exit as driven by the presence of a more lucrative outside option to the firm's owner rather than a flow fixed cost of operation. Denote by c_f the value to the owner in case she shuts down the firm. The problem facing the coalition is to optimally separate, exit and post vacancies such as to maximize the joint coalition value,

$$\begin{aligned}
\rho\Omega(z,n) &= \max_{v\geq 0} \quad y\left(z,n\right) - \delta n\left[\Omega_n(z,n) - U\right] \\
&+ \quad q\left[\phi\left[\Omega_n(z,n) - U\right] + (1-\phi)\int \max\left\{\Omega_n\left(n,z\right) - \Omega_n\left(n',z'\right),0\right\} dG\left(n',z'\right)\right] v \\
&- \quad c\left(v,n\right) \\
&+ \quad \mu(z)\Omega_z(z,n) + \frac{\sigma^2(z)}{2}\Omega_{zz}(z,n) \\
s.t. & \\
\Omega(z,n) &\geq nU + c_f \\
\Omega_n(z,n) &\geq U
\end{aligned} \tag{D1}$$

D.1 Simplifying the problem of the coalition

We can rewrite the term under the integral sign in (D1) to integrate directly over $\Omega_n' = \Omega_n\left(n',z'\right)$

$$\int_U^{\Omega_n(n,z)} \left[\Omega_n\left(n,z\right) - U\right] - \left[\Omega_n' - U\right]\, dG\left(\Omega_n'\right)$$

where we used the fact that the lower bound of the support must be U, since $\Omega_n' \geq U$ and the upper bound is given by the fact that a firm only hires if $\Omega_n' \leq \Omega_n\left(n,z\right)$, and we added and subtracted U in the integrand. Since $\Omega_n' \in \left[U, \Omega_n\left(n,z\right)\right]$ implies $\Omega_n' - U \in \left[0, \Omega_n\left(n,z\right) - U\right]$, we can integrate over $\Omega_n' - U$ and adjust the bounds

$$\int_0^{\Omega_n(n,z)-U} \left[\Omega_n\left(n,z\right) - U\right] - \left[\Omega_n' - U\right]\, dG\left(\Omega_n' - U\right)$$

We can hence restate the problem (D1) as

$$
\begin{aligned}
\rho\Omega(z,n) &= \max_{v\geq 0} \quad y(z,n) - \delta n\left[\Omega_n(z,n) - U\right] \qquad \text{(D2)}\\
&+ q\left[\phi\left[\Omega_n(z,n) - U\right] + (1-\phi)\int_0^{\Omega_n(n,z)-U} \left[\Omega_n(n,z) - U\right] - \left[\Omega_n' - U\right] dG\left(\Omega_n' - U\right)\right]v\\
&- c(v,n)\\
&+ \mu(z)\Omega_z(z,n) + \frac{\sigma^2(z)}{2}\Omega_{zz}(z,n)\\
s.t.&\\
\Omega(z,n) &\geq nU + c_f\\
\Omega_n(z,n) &\geq U
\end{aligned}
$$

Let $S(z,n)$ be the *surplus* of the coalition, $S(z,n) = \Omega(z,n) - nU$. Note that $S_z(z,n) = \Omega_z(z,n)$, $S_{zz}(z,n) = \Omega_{zz}(z,n)$ and $S_n(z,n) = \Omega_n(z,n) - U$. Substituting this in problem (D2),

$$
\begin{aligned}
\rho S(z,n) &= \max_{v\geq 0} \quad y(z,n) - \delta n S_n(z,n)\\
&+ q\left[\phi S_n(z,n) + (1-\phi)\int_0^{S_n(z,n)} S_n(z,n) - S_n'\, dG\left(S_n'\right)\right]v\\
&- c(v,n)\\
&+ \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2}S_{zz}(z,n) - \rho n U\\
s.t.&\\
S(z,n) &\geq c_f\\
S_n(z,n) &\geq 0
\end{aligned}
$$

Integrate by parts the expected value of a vacancy conditional on meeting an employed worker

$$
\begin{aligned}
\int_0^{S_n(z,n)} \left[S_n(z,n) - S_n'\right] dG\left(S_n'\right) &= \left[S_n(z,n) - S_n'\right] G\left(S_n'\right)\Big|_0^{S_n(z,n)} + \int_0^{S_n(z,n)} G\left(S_n'\right) dS_n'\\
&= \left[S_n(z,n) - S_n(z,n)\right] G\left(S_n(z,n)\right)\\
&- \left[S_n(z,n) - 0\right] G(0) + \int_0^{S_n(z,n)} G\left(S_n'\right) dS_n'
\end{aligned}
$$

The second term on the second line is equal to zero since the constraint on the firms' problem $S_n(z,n) \geq 0$ implies that the distribution of marginal surpluses at other firms must also be zero at zero

$$
\int_0^{S_n(z,n)} \left[S_n(z,n) - S_n'\right] dG\left(S_n'\right) = \int_0^{S_n(z,n)} G\left(S_n'\right) dS_n'
$$

Define $\mathcal{G}(x)$ as the integral of the cdf G: $\mathcal{G}(x) = \int_0^x G(u)\, du$

$$\int_0^{S_n(z,n)} \left[S_n(z,n) - S_n'\right] dG\left(S_n'\right) = \mathcal{G}\left(S_n(z,n)\right)$$

Substituting this into the Bellman equation

$$\begin{aligned}
\rho S(z,n) &= \max_{v \geq 0} \quad y(z,n) - \delta n S_n(z,n) \\
&+ \quad q\left[\phi S_n(z,n) + (1-\phi)\,\mathcal{G}\left(S_n(z,n)\right)\right] v \\
&- \quad c(v,n) \\
&+ \quad \mu(z) S_z(z,n) + \frac{\sigma^2(z)}{2} S_{zz}(z,n) - \rho n U \\
s.t. & \\
S(z,n) &\geq c_f \\
S_n(z,n) &\geq 0
\end{aligned}$$

We assume that the vacancy cost satisfies $c(v,n) = \bar{c}\left(\frac{v}{n}\right) v$, where $\bar{c}$ is iso-elastic with elasticity γ. Define the function $\mathcal{H}(x)$ by $\mathcal{H}(x) = q\left[\phi x + (1-\phi)\,\mathcal{G}(x)\right]$. Substituting this into problem (D2)

$$\begin{aligned}
\rho S(z,n) &= \max_{v \geq 0} \quad y(z,n) - \delta n S_n(z,n) \\
&+ \quad \mathcal{H}\left(S_n(z,n)\right) v - c(v,n) \\
&+ \quad \mu(z) S_z(z,n) + \frac{\sigma^2(z)}{2} S_{zz}(z,n) - \rho n U
\end{aligned}$$

Since $\bar{c}(v/n)$ is iso-elastic in (v/n), $c_v(v,n) = (\gamma+1)\,\bar{c}\left(\frac{v}{n}\right)$.[22] Along with the first order condition $c_v(v,n) = \mathcal{H}\left(S_n(z,n)\right)$, this implies

$$c(v,n) = \bar{c}\left(\frac{v}{n}\right) v = \frac{1}{\gamma+1} c_v(v,n)\, v = \frac{1}{\gamma+1} \mathcal{H}\left(S_n(z,n)\right) v$$

Therefore the total value of vacancy posting is

$$\begin{aligned}
\mathcal{H}\left(S_n(z,n)\right) v - c(v,n) &= \frac{\gamma}{\gamma+1} \mathcal{H}\left(S_n(z,n)\right) v \\
\mathcal{H}\left(S_n(z,n)\right) v - c(v,n) &= \frac{\gamma}{\gamma+1} \mathcal{H}\left(S_n(z,n)\right) \left(\frac{v}{n}\right) n
\end{aligned}$$

22

$$c_v = \bar{c}' \frac{v}{n} + \bar{c} = \left(\frac{\bar{c}'}{\bar{c}} \frac{v}{n} + 1\right) \bar{c} = (\gamma+1)\,\bar{c}$$

Letting $\bar{c}\left(\frac{v}{n}\right)=\frac{\kappa}{1+\gamma}\left(\frac{v}{n}\right)^{\gamma}$ and using $\bar{c}\left(\frac{v}{n}\right)=\frac{1}{\gamma+1}\mathcal{H}\left(S_n\left(z,n\right)\right)$ then

$$\frac{v}{n}=\kappa^{-1/\gamma}\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{1}{\gamma}}$$

and

$$\mathcal{H}\left(S_n\left(z,n\right)\right)v-c\left(v,n\right)=\frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{\gamma+1}{\gamma}}n$$

Substituting this into the Bellman equation

$$\begin{aligned}\rho S\left(z,n\right) &= y\left(z,n\right)-\delta nS_n\left(z,n\right)\\ &+ \frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{\gamma+1}{\gamma}}n\\ &+ \mu(z)S_z(z,n)+\frac{\sigma^2(z)}{2}S_{zz}(z,n)-\rho nU\end{aligned}$$

Collecting terms and recognizing that $\rho U=b$,

$$\begin{aligned}\rho S\left(z,n\right) &= y\left(z,n\right)-bn\\ &+ \left[\frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\frac{\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{\gamma+1}{\gamma}}}{S_n\left(z,n\right)}-\delta\right]S_n\left(z,n\right)n\\ &+ \mu(z)S_z(z,n)+\frac{\sigma^2(z)}{2}S_{zz}(z,n)\end{aligned}\tag{D3}$$

subject to

$$\begin{aligned}S(z,n) &\geq c_f\\ S_n(z,n) &\geq 0\\ \mathcal{H}\left(S_n\left(z,n\right)\right) &= q\left[\phi S_n\left(z,n\right)+\left(1-\phi\right)\mathcal{G}\left(S_n\left(z,n\right)\right)\right]\\ \mathcal{G}\left(S_n\left(z,n\right)\right) &= \int_0^{S_n(z,n)}G\left(s\right)ds\end{aligned}$$

D.2 Algorithm

The algorithm consists of three steps, implemented in MATLAB called from master file `MAIN.m`.

Step 0: Construct an initial guess. Start by constructing a $n_z\times n_n$ grid for log productivity and log size. Note that the derivative of surplus w.r.t. the log of size satisfies $S_N N=S_n$. Let $\mathbf{F}=y(z,n)-bn$ denote the stacked $(n_z*n_n)\times 1$ vector of flow payoffs on this grid. Guess an initial surplus $\mathbf{S}^0$ on this grid (a $(n_z*n_n)\times 1$ column vector); a distribution of firms over productivity and size $\mathbf{h}^0$ (a $(n_z*n_n)\times 1$ column vector); aggregate finding rates q^0 and λ^0; and an efficiency-weighted share of unemployed searchers, θ^0. Construct marginal surplus. Set the value of the outside option to firm owners to $c_f = S(1,1)$, and construct exit regions, separation regions and the vacancy policy. File `InitialGuess.m` does this.

Step I: Iterate to convergence the coalition's problem for given aggregate states. For $t \geq 1$, given q^{t-1}, θ^{t-1}, h^{t-1} and $\mathbf{S}^{t-1}$, solve the coalition's problem to update the coalition value to $\mathbf{S}^t$. The solution to the coalition's surplus function is obtained in an inner iteration τ. Denote by $\mathbf{S}^{t,\tau}$ the surplus in outer iteration t during inner iteration τ, initiated with $\mathbf{S}^{t,0} = \mathbf{S}^t$; $\mathbf{T}_n(z,n)$ a $(n_z * n_n) \times (n_z * n_n)$ matrix such that $\mathbf{S}_n^{t,\tau} = \mathbf{T}_n(z,n)\mathbf{S}^{t,\tau}$, where $\mathbf{S}_n^{t,\tau}$ is the stacked $(n_z * n_n) \times 1$ vector of derivatives of S w.r.t. n during outer iteration t and inner iteration τ; $\mathbf{T}_z$ a $(n_z * n_n) \times (n_z * n_n)$ matrix such that $\mathbf{S}_z^{t,\tau} = \mathbf{T}_z\mathbf{S}^{t,\tau}$, where $\mathbf{S}_z^{t,\tau}$ is the stacked $(n_z * n_n) \times 1$ vector of derivatives of S w.r.t. z during outer iteration t and inner iteration τ; and $\mathbf{T}_{zz}$ a $(n_z * n_n) \times (n_z * n_n)$ matrix such that $\mathbf{S}_{zz}^{t,\tau} = \mathbf{T}_{zz}\mathbf{S}^{t,\tau}$, where $\mathbf{S}_{zz}^{t,\tau}$ is the stacked $(n_z * n_n) \times 1$ vector of second derivatives of S w.r.t. z during outer iteration t and inner iteration τ. Note that the matrix $\mathbf{T}_n(z,n)$ depends on (z,n) in the sense that the approximation is done either forward or backward depending on the endogenous drift for n at (z,n) (note that the drift of and innovations to z are independent of (z,n)). Within each outer iteration t, we iteratively update $\mathbf{S}^{t-1,\tau}$ for $\tau \geq 1$ following equation (D3) based on

$$\left[\left(\rho + \frac{1}{\Delta}\right)\mathbb{1} - \left[\frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\frac{\mathcal{H}\left(\mathbf{S}_n^{t-1,\tau-1}\right)^{\frac{\gamma+1}{\gamma}}}{\mathbf{S}_n^{t-1,\tau-1}} - \delta\mathbb{1}\right] .* \; T_n(z,n) - \mu T_z - \frac{\sigma^2}{2}T_{zz}\right]\mathbf{S}^{t-1,\tau} = \mathbf{F} + \frac{1}{\Delta}\mathbf{S}^{t-1,\tau-1}$$

where Δ is the step size, $.*$ denotes the element-by-element product, and $\mathcal{H}\left(\mathbf{S}_n^{t-1,\tau-1}\right)^{\frac{\gamma+1}{\gamma}} / \mathbf{S}_n^{t-1,\tau-1}$ is a $(n_z * n_n) \times (n_z * n_n)$ matrix constructed using the previous iteration's derivative of S stacked $(n_z * n_n)$ times in the column dimension. The step size cannot be too large for the problem to converge. These iterations are performed by iterating on τ until convergence by file `IndividualBehavior.m`, and the solution is assigned as the updated $\mathbf{S}^t$. We also obtain from the converged solution the updated separation, exit and a vacancy policies.

Step II: Iterate to convergence the aggregate states for given individual behavior. Given updated individual behavior in outer iteration t, obtain through iteration in an inner loop τ the distribution of firms $\mathbf{h}^t$, the aggregate meeting rates q^t and λ^t, the share of unemployed searchers θ^t, the distribution of workers over marginal surplus $\mathbf{G}^t$, and the distribution of vacancies over marginal surplus $\mathbf{F}^t$. File `AggregateBehavior.m` proceeds to do this in four steps.

Initiate each aggregate object with the previous outer iteration solution, $\mathbf{x}^{t-1,0} = \mathbf{x}^{t-1}$. Then:

Step II-a. Update the distribution of workers over marginal surplus to $\mathbf{G}^{t-1,\tau}$ given a distribution of firms $\mathbf{h}^{t-1,\tau-1}$ and marginal surplus $\mathbf{S}_n^t$, where the latter was obtained in **Step I** above. This is done by file `CdfG.m`.

Step II-b. Update the distribution of vacancies over marginal surplus $\mathbf{F}^{t-1,\tau}$ given a distribution of firms $\mathbf{h}^{t-1,\tau-1}$, the vacancy policy $\mathbf{v}^t$ and the ranking of firms in marginal surplus space. This is done by file `CdfF.m`.

Step II-c. Update the finding rates $q^{t-1,\tau}$, $\lambda^{t-1,\tau}$ and $\theta^{t-1,\tau}$ that is consistent with the vacancy policy $\mathbf{v}^t$ and the distribution of firms $\mathbf{h}^{t-1,\tau-1}$. This is done by file `HazardRates.m`.

Step II-d. Given $\mathbf{G}^{t-1,\tau}$, $\mathbf{F}^{t-1,\tau}$, $q^{t-1,\tau}$, $\lambda^{t-1,\tau}$ and $\theta^{t-1,\tau}$, update the distribution of firms $\mathbf{h}^{t-1,\tau}$ following the Kolmogorov forward equation in steady-state. This is executed by file `Distribution.m`.

Iterate over the four sub-steps *Step II-a–Step II-d* until convergence and assign the updated aggregate states q^t, λ^t, θ^t and $\mathbf{h}^t$. We subsequently return to step **Step I** and iterate on step **Step I–Step II** until both the surplus function and the aggregate states have converged.