# Technological Diversification*

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**Abstract**

Economies at early stages of development are often shaken by abrupt changes in growth rates, whereas in advanced economies growth rates tend to be relatively stable. To explain this pattern, we propose a theory of technological diversification. Production makes use of different input varieties, which are subject to imperfectly correlated shocks. Technological progress takes the form of an increase in the number of varieties, raising average productivity. In addition, the expansion in the number of varieties in our model provides diversification benefits against variety-specific shocks and it can hence lower the volatility of output growth. Technological complexity evolves endogenously in response to profit incentives. The decline in volatility thus arises as a by-product of firms' incentives to increase profits and is hence a likely outcome of the development process. We quantitatively asses the predictions of the model in light of the empirical evidence and find that for reasonable parameter values, the model can generate a decline in volatility with the level of development comparable to that in the data.

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# 1 Introduction

Economies at early stages of the development process are often shaken by abrupt changes in growth rates. In his influential paper, Lucas (1988) brings attention to this fact, noting that "within the advanced countries, growth rates tend to be very stable over long periods of time," whereas within poor countries "there are many examples of sudden, large changes in growth rates, both up and down."

Motivated by this empirical observation, this paper proposes an endogenous growth model of technological diversification. The key idea of the model is that firms using a large variety of inputs can mitigate the impact of shocks affecting the productivity of individual inputs. This takes place through two channels. First, with a larger number of inputs, each individual input matters less in production, and productivity becomes less volatile by the law of large numbers. Second, whenever a shock hits a particular input, firms can adjust the use of the other inputs to partially offset the shock. Both channels make the productivity of firms using more sophisticated technologies less volatile. Since firms in richer economies tend to rely on technologies involving a richer menu of inputs, richer countries will also tend to be less volatile.

The starkest example of this mechanism of technological diversification is offered by a comparison of an economy using only labour and an economy using labour and capital. Under standard assumptions on technology the latter will tend to be more productive on a per-capita basis. Our point is that it will also be less volatile. In particular, any shock that reduces the supply of labour (such as a general strike, an epidemic, etc.) will have a bigger negative impact on the economy that does not have scope to substitute labour with capital. Or, to think of a currently more realistic example, consider leading-edge steel producers that have the capacity to process iron ore of a range of qualities as compared to more basic producers who can only accept high-quality ores as input. Clearly the former are more productive, and, in addition, they should be less susceptible to shocks to the (global or local) supply of high-quality iron ore.

The stabilizing virtues of technological diversification are currently much in evidence in the debate over energy policy. The recent increase in oil prices has led to overwhelming bipartisan support in the U.S. for the H-prize Act of June 2007, which seeks to incentivise "achievements in overcoming scientific and technical barriers associated with hydrogen energy" in order "to free [the country] from its dependence on foreign oil."[1] It has also led to increased demand for vehicles that can run on a range

[1] The first quotation comes from the Act text itself. The second comes from its sponsor's speech at the House of Representatives; the Act was passed with 408 Ayes and 8 Nays.

of energy sources beyond gas.[2]

Our model builds on the seminal contributions by Romer (1990) and Grossman and Helpman (1991) and characterizes technological progress as an expansion in the number of input varieties.[3] The number of varieties evolves endogenously in response to producers' incentives to add to the range of inputs they use, and increases in the number of varieties raise the average level of productivity. What is new here is that each input variety can be hit by a productivity shock, so that the expansion in the number of varieties can provide diversification benefits, and hence reduces the level of volatility.[4] In other words, the reduction in volatility in the model arises as a likely by-product of firms' incentives to increase productivity. As such, our model highlights a hitherto overlooked implication of expanding-variety growth models, which makes them suitable to explain the decline in volatility that accompanies the development process.

We say "suitable to explain" because, interestingly, once technological diversification is embedded in an endogenous growth model with multiple firms, it is possible to generate examples where volatility and development do not necessarily move in opposite directions. Heuristically, in our model the growth process determines the evolution of both the number of varieties used by the typical firm, and the contribution of each variety to aggregate output; complete technological diversification in the economy is achieved (conditional on the number of different varieties) when the contributions to output of all varieties are equalized, since this minimizes the reliance of the economy on each individual variety and hence the potential impact of a variety-specific productivity shock. Roughly speaking, the number of varieties used by the typical firm is what matters most for development, while the extent of technological diversification is what matters most for (reducing) aggregate volatility, and theoretically it is possible for them to move in different directions. Intuitively, however, the two should move together most of the time, and this is indeed the case in virtually all our numerical experiments. This is because the introduction of a *new* variety in the economy, *ceteris paribus*, always increases the level of development, and raises the degree of technological diversification by reducing the contribution to output of previously existing varieties (thus lowering volatility). We also find that for reasonable parameter values, the model

[2] The Economist, 05/06/06, page 52, "Alternative Energy: Canola and Soya to the Rescue" and "The Economist, 12/05/07, page 47, Craze for Maize: Ethanol is rapidly transforming life in Iowa and the rest of the corn belt." To be sure, increased concern with global warming has also played a part in this trend.

[3] See also Barro and Sala-i-Martin (2004) for a comprehensive formalization and discussion of expanding-variety models.

[4] Shocks in this model are variety-specific and to the extent that the varieties are used by a positive measure of firms, they lead to aggregate volatility.

can yield a decline in volatility with development quantitatively comparable to that in the data.

Previous theoretical work on the relation between volatility and development, including Greenwood and Jovanovic (1990), Saint-Paul (1992), Obstfeld (1994), and Acemoglu and Zilibotti (1997), has focused on *financial*—as opposed to *technological*—diversification. These models feature an inherent trade-off between productivity and risk at the firm-level: Firms must choose between low-return but safe activities and high-return but risky ones. Firms in financially underdeveloped countries do not have the facility to pool risks, so risk-averse entrepreneurs minimize firm-level risk by choosing low productivity projects. In financially developed countries, risks can be pooled and hence entrepreneurs undertake high-return and high-risk projects. This leads to an increase in firm-level volatility with development. Aggregate volatility may still be lower in developed countries if financial deepening facilitates the creation of new financial diversification opportunities across firms, as is the case in Greenwood and Jovanovic (1990) and Acemoglu and Zilibotti (1997).

Unlike existing models, the expanding-variety model we propose posits no trade-off between productivity and risk at the firm level. Indeed our point is that there are technological reasons to expect the adoption of a new variety to concurrently lead to an increase in productivity and a decline in volatility. Hence, preferences towards risk, which are crucial in models of "financial diversification," play no role in our story, where firms are uniquely concerned with profit maximisation. Similarly, and perhaps most importantly, the process of technological diversification does not hinge on the patterns of financial development.

These theoretical differences lead to important differences in empirical implications. First, models of financial diversification predict an increase in firm-level volatility with the level of development, while our model predicts a decline in firm-level volatility.[5] The decline in firm-level volatility finds support in recent work by Davis, Haltiwanger, Jarmin and Miranda (2006), who document that in the US, over time, privately held firms have experienced a substantial decline in volatility; the authors furthermore show that the decline in aggregate volatility in the US has been overwhelmingly driven by the decline in firm-level volatility (and not by the aggregation of highly volatile firms).[6]

[5] The unit of analysis in these models can be construed as a *sector* rather than a *firm*. From a theoretical point of view, interpreting it as a firm is more appealing, as the optimizing agent will likely operate at the firm level. Furthermore, the sectoral interpretation is challenged by the finding that sectoral volatility declines with productivity and the overall level of development. (Koren and Tenreyro, 2007).

[6] Comin and Philippon (2005) had previously documented that publicly traded firms in the US experienced an increase in volatility during the same period. However, as stressed by Davis et al. (2006), publicly traded firms are a small fraction of all privately held firms. Furthermore, since a

A second testable difference is that in models of financial diversification the decline in aggregate volatility with development is brought about by financial deepening. In our model, the decline in volatility takes place independently of the level of financial deepening. As we show in the next section, this implication is corroborated by the evidence. The strong negative correlation between volatility and development takes place at all levels of financial depth. Put differently, even controlling for the level of financial development, there remains a strong negative correlation between volatility and development that needs explanation. Even more importantly, while in cross-country data volatility and financial deepening are negatively correlated, this correlation vanishes once one controls for country-specific fixed effects, casting some doubt on the ability of a financial channel to explain the time-series evidence.[7] While we view both margins of diversification for the firm, financial and technological, as complementary and empirically plausible, our model will focus exclusively on the second.[8]

As mentioned, our model posits no trade-off between productivity and volatility at the firm level. The absence of trade-off can be substantiated by two pieces of evidence: First, more productive and larger firms appear to be also less volatile, a result we document in the next section. And second, as shown in Koren and Tenreyro (2007), countries at early stages of development tend to specialize in low-productivity, high-risk sectors, whereas the opposite pattern is observed at later stages; in other words, the development process is characterized by a move towards both more productive and safer sectors.[9]

It is also important to distinguish our mechanism of technological diversification from standard arguments concerning sectoral diversification, namely that developing countries should reduce their reliance on cash crops or natural resources in order to hedge against fluctuations in these commodities' prices. First, our model concerns the diversification of inputs, not the diversification of outputs. Second, and most impor-

majority of firms in most countries are privately held, the evidence from Davis et al. (2006) seems more relevant to our model, which aims at understanding the differences in aggregate and firm-level volatility between developed and developing countries.

[7] In contrast, the correlation between volatility and development is strong both in cross-sectional and within-country analyses.

[8] Technological diversification is also complementary to other other finance-related mechanisms emphasized in the literature. In particular, shocks can be amplified by introducing financial frictions, a task we do not undertake in the interest of clarity and simplicity. For models with financial frictions, see, among others, Bernanke and Gertler (1990), Kiyotaki and Moore (1997), Aghion, Angeletos, Banerjee and Manova (2004).

[9] In departing from a necessary trade-off between productivity and volatility, our paper is closer to Kraay and Ventura (2007), though the mechanisms are completely different: in their model, the key idea is that in the event of a shock, terms of trade respond more countercyclically in rich countries than in poor countries.

tantly, sectoral diversification is usually associated with a move away from comparative-advantage, so it tends to reduce (average) income. Instead, technological diversification chiefly occurs as a by-product of strategies whose main aim is to increase average income.[10]

The paper begins with a summary of empirical regularities that motivate our model and differentiate it from models emphasizing financial diversification. Section 3 presents the model of technological diversification and derives its implications for aggregate dynamics. Section 4 presents a quantitative analysis of the model. Section 5 offers concluding remarks. All proofs are in the Appendix.

## 2 Empirical Motivation

This Section summarizes and discusses four empirical regularities that motivate the model of technological diversification we present.

**Regularity 1.** GDP volatility declines with development, both in the cross section and for a given country over time.

The negative association between aggregate volatility and the level of development is one of the stylized facts in the macro-development literature and the starting motivation of this paper. The relation is illustrated in the left hand-side panel of Figure 1, which plots the (log) level of volatility, measured as the standard deviation of annual growth rates over non-overlapping decades from 1960 through 2000, against the average (log) level of real GDP per capita of the decade.[11] The graph also shows the linear regression line together with the 95-percent confidence-band intervals. The second panel of Figure 2 plots the same variables after controlling for country-specific fixed effects and shows that for a given country over time, growth and changes in volatility are also negatively correlated.[12]

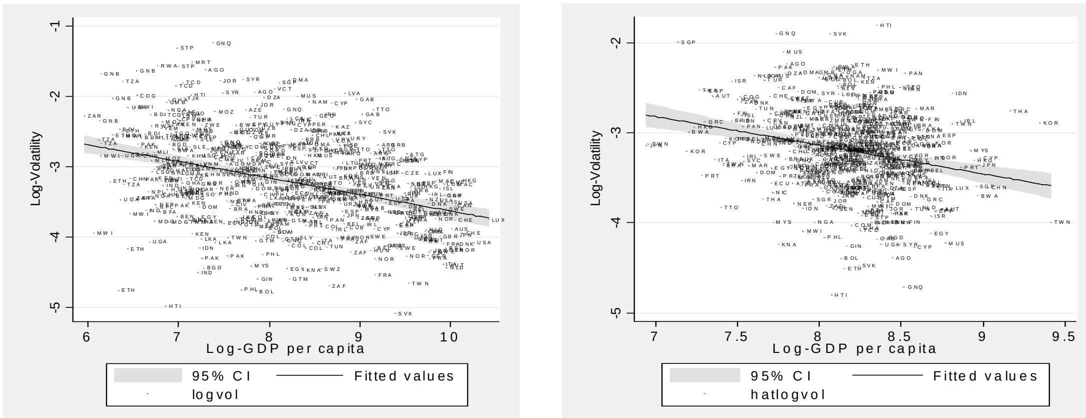
[10] In fact, sectoral diversification as a hedging strategy is dominated by financial hedging on commodity-futures markets. As discussed, no such (better) substitute exists for technological diversification.

[11] The data come from the World Bank's *World Development Indicators*.

[12] In related work, Ramey and Ramey (1995) study the link between volatility and growth. We focus instead on the links between volatility and *development* or between *changes* in volatility and growth, to be consistent with the predictions of the model we later develop.

Figure 1: Volatility and Development

Cross-Sectional Relation | Within-Country Relation




Note: The plots show log volatility (standard deviation of annual growth rates over non-overlapping decades from 1960 through 2000) against the average (log) real GDP per capita of the decade, without controlling for country- fixed effects (left panel) and controlling for fixed effects (right panel). Regression lines and 95% confidence intervals also displayed.

The negative relationship between volatility and productivity is of course also a within-industry phenomenon. Figure 2 provides a graphical illustration of this for agriculture. It displays the volatility of wheat yield (calculated as the standard deviation of annual yield changes) of the 20 biggest wheat producers against their level of GDP per capita. As the plot shows, yield volatility declines sharply with the level of development.[13]

Our model of technological diversification is able to generate a negative correlation between volatility and development as countries with a larger number of varieties are more productive and, in general, better diversified across varieties. In other words, the high volatility at early stages of development results from the relatively low number of varieties used in the production process.[14]

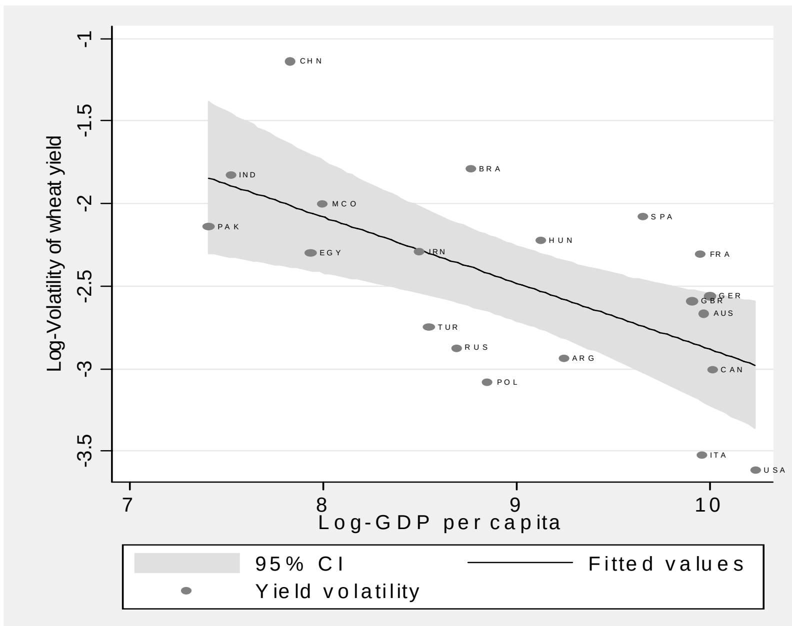
In the case of agriculture, for example, growing wheat with only land and labour as inputs renders the yield vulnerable to idiosyncratic shocks. In contrast, using land

[13] This remains true if we control for differences in climate across countries, including the volatility of rainfall and temperature.

[14] Various empirical studies document the slow or delayed process of adoption of varieties in developing countries. For example, Caselli and Coleman (2001) find that the adoption of computers depends crucially on the level of development of the country and Caselli and Wilson (2004) show that this result extends to a broader set of more sophisticated equipment. In related work, Comin and Hobijn (2004) find that most innovations originate in developed countries and spread only gradually to less-developed countries.

and labour together with artificial irrigation, different varieties of fertilizers, pesticides, etc., can make wheat-growing not only more productive on average but also less risky, because farmers have more options to substitute failing, unavailable, or simply temporarily expensive inputs. Note that agricultural technology varies substantially with development. For example, of the top 20 wheat producers, India uses 2.3 tractors per 1,000 acres of arable land; this number is 128.8 for Germany. Fertilizer use also varies hugely. India uses 21.9 tons of nitrogenous fertilizers per 1,000 acres; Germany uses 183.8 tons.[15]

Figure 2. Wheat Yield Volatility and Development




Note: The Figure plots the volatility of wheat yield (standard deviation of annual yield changes) of the 20 biggest wheat producers against (log) real GDP per capita. OLS regression line and 95% confidence intervals also shown. *Source*: FAOSTAT 2005.

**Regularity 2.** The negative association between aggregate volatility and development takes place at all levels of financial development.

The relation between aggregate volatility and development holds at different levels of financial development, measured, as is standard, by the (log) ratio of private credit to GDP.[16] This is illustrated in Figure 3, where we split the level of financial development

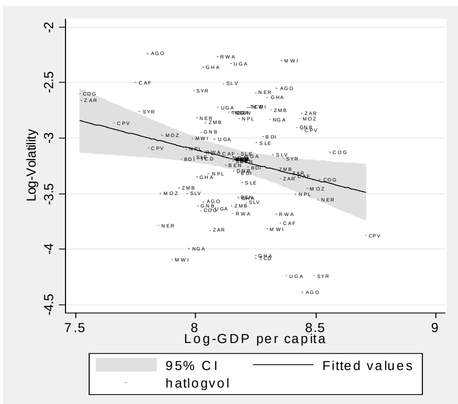
[15] Food and Agriculture Organization of the United Nations, FAOSTAT Yearbook 2005.

[16] Data on private credit over GDP come from the World Bank's *World Development Indicators*. This is the most widely used measure of financial development, because it is available for the broadest set of countries.

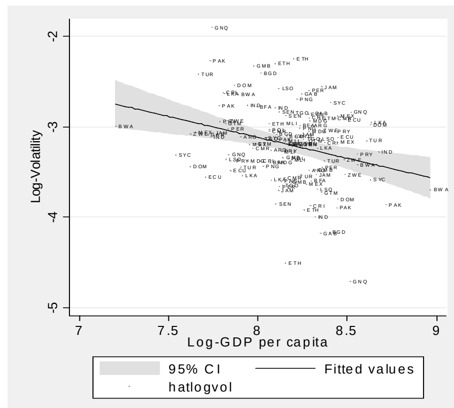
into different quartiles. The graphs show that the decline of volatility with development is not sensitive to the level of financial development of the country. That is, controlling by financial development, there is still a strong negative association between volatility and development that needs explanation.

Figure 3: Volatility and Development by Financial Development Quartile

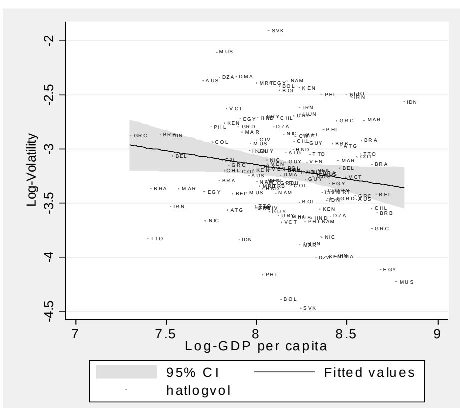
First Quartile



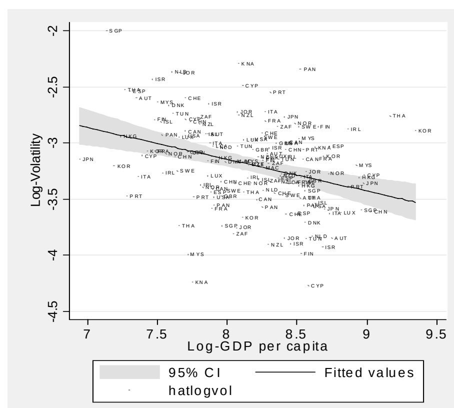

Second Quartile




Third Quartile




Fourth Quartile



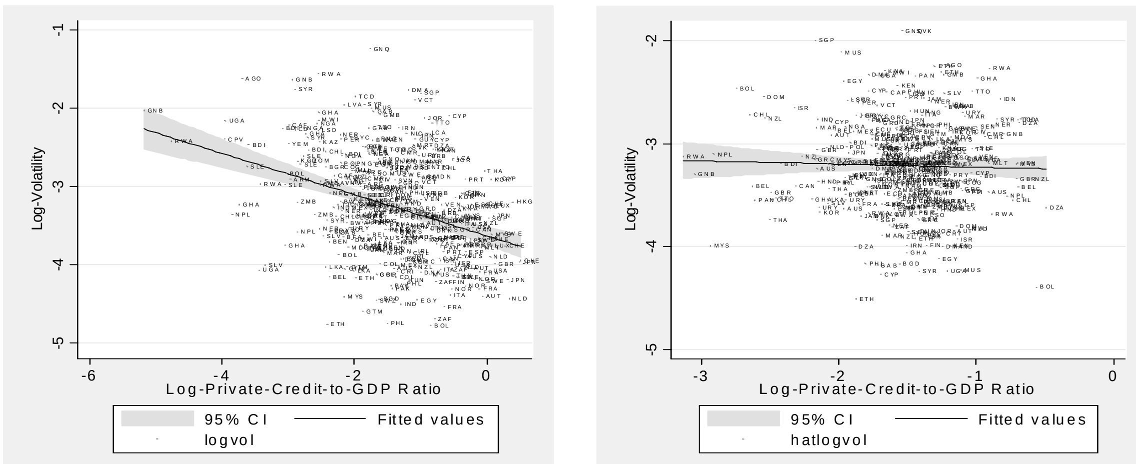

Note: The plots show log volatility (standard deviation of annual growth rates over non-overlapping decades from 1960 through 2000) against the average (log) real GDP per capita of the decade, for different quartiles of financial development (private credit over GDP). Regression lines and 95% confidence intervals also displayed.

**Regularity 3.** The correlation between aggregate volatility and financial development is negative in the cross-section, but nil in the time-series evidence.

Figure 4 plots the (pooled) cross-sectional and within-country relations between volatility and the (log) level of financial development (as measured before). As shown in the plots, there is a strong negative correlation in the cross-section. The correlation, however, vanishes once country-specific fixed effects are controlled for. This finding suggests that the decline in volatility for a given country over time cannot be explained in a statistical sense by higher levels of financial development, calling for alternative channels.

Figure 4: Volatility and Financial Development

Cross Sectional Relation | Within-Country Relation




Note: The Figures plot log volatility (standard deviation of annual growth rates over non-overlapping decades from 1960 through 2000) against the average (log) level of financial development of the decade (measured as private credit over GDP, without controlling for country- fixed effects (left panel) and controlling for fixed effects (right panel). Regression lines and 95% confidence intervals also displayed.

We summarize these correlations in Table 1. The first two columns show the coefficients from a regression of (log) volatility on (log) real GDP per capita, excluding and including fixed effects. The coefficients are statistically significant at the 1 percent level. The third and fourth columns show the corresponding results when volatility is regressed on the (log) ratio of private credit to GDP. As anticipated, the cross sectional relation is negative; however, once fixed effects are included, the estimated elasticity is both statistically and economically insignificant. Finally the last two columns show the regression results when both variables are included in the regression. Volatility is strongly (and negatively) associated with per capita GDP, but it shows little or no (partial) correlation with the level of financial development.

Table 1. Volatility, Development, and Finance

| | Dependent Variable: Standard Deviation of Growth Rates | | | | | |
|---|---|---|---|---|---|---|
| GDP per capita (constant PPP $) | -0.2319*** | -0.3008*** | | | -0.1689*** | -0.3153*** |
| | [0.0318] | [0.0622] | | | [0.0454] | [0.1169] |
| Private Credit / GDP | | | -0.1468*** | -0.0198 | -0.0711* | 0.0124 |
| | | | [0.0542] | [0.0489] | [0.0380] | [0.0539] |
| Country Fixed Effects | No | Yes | No | Yes | No | Yes |
| Observations | 585+ | 585+ | 403 | 403 | 403 | 403 |
| R-squared | 0.13 | 0.56 | 0.09 | 0.60 | 0.13 | 0.62 |

Note: All variables are in logs. The dependent variable is measured as the standard deviation of annual growth rates over non-overlapping decades from 1960 to 2000. The regressors are computed at their mean values over the decade. Clustered (by country) standard errors in brackets. * Significant at 10%; ** significant at 5%; *** significant at 1%. + When the sample is constrained to the 403 bservations for which private credit/GDP is available, the corresponding coefficients in the first and second columns are 0.221 and 0.307, significant at the 1% level.

Regularities 2 and 3 motivate our search for alternative explanations (beyond financial development) that can yield the decline in volatility with development. In the model, hence, we shut down the financial-development channel to focus exclusively on technological diversification.

**Regularity 4.** More productive firms are less volatile. The decline in aggregate volatility coincides with the decline in firm-level volatility.

Firm-level evidence suggests that there is no trade-off between lower volatility and productivity. More concretely, productivity and volatility in US firm-level data are strongly negatively correlated.[17] This is illustrated in Table 2, which shows the coefficients from a regression of (log) volatility of sales growth on average size (employment) and productivity (sales per worker) for 9000 Compustat firms. Volatility is calculated for non-overlapping decades from 1950 through 1999. Both productivity and size are negatively correlated with firm-level volatility.[18] The negative correlation remains strong even if we include firm-fixed effects to consider within-firm variation only; in other words, firms becoming more productive also become less volatile.

This empirical observation motivates an important feature of our model of technological diversification that differentiates it from models of financial diversification: Unlike the latter, our model posits no necessary trade-off between lower volatility and productivity at the firm level. As a result, and consistent with the data, in our model more productive (and larger) firms are also relatively less volatile.

[17] As usual, the paucity of firm-level data for less developed countries motivates the focus on US data.

[18] The negative correlation between firm-level volatility and size has been documented in an early study by Hymer and Pashigian (1962).

A second important empirical observation is that, as shown by Davis, et al. (2006), the decline in aggregate volatility in the US has been overwhelmingly driven by a decline in firm-level volatility and *not* by the aggregation of increasingly more volatile firms displaying lower correlation in their performance.

This is another crucial aspect of our model, which distinguishes it from models of financial diversification. The decline in aggregate volatility with development coincides with (and in fact is driven by) the decline in firm-level volatility. In models of financial diversification, instead, the decline in aggregate volatility with development takes place together with an increase in firm-level volatility (and a necessary decrease in cross-firm correlation).

Table 2. Firm-Level Productivity and Volatility

| | Dependent Variable: Standard Deviation of Growth | | | |
|---|---|---|---|---|
| Sales per worker | -0.115*** | | -0.137*** | -0.136*** |
| | [0.008] | | [0.007] | [0.017] |
| Employment | | -0.192*** | -0.194*** | -0.198*** |
| | | [0.002] | [0.002] | [0.009] |
| Firm Fixed Effects | No | No | No | Yes |
| Decade Fixed Effects | Yes | Yes | Yes | Yes |
| Observations | 25408 | 25408 | 25408 | 25408 |
| R-squared | 0.06 | 0.24 | 0.26 | 0.60 |

Note: All variables are in logs. The dependent variable is measured as the standard deviation of annual sales growth rates over non-overlapping decades from 1960 to 2000. The regressors are computed at their mean values over the decade. Clustered standard errors in brackets. *Significant at 10%; **significant at 5%; *** significant at 1%.

The empirical regularities listed in this Section motivate the model that we develop next.

# 3 A Model of Technological Diversification

This Section starts by first formalizing the intuition of the model for a single firm in a static setup. It then presents the dynamic, stochastic, multi-firm model.

## 3.1 Production technology: Formalizing the intuition in a static setup

Consider the following production process for a firm. Output is produced by combining a variety of inputs in a constant-elasticity-of-substitution (CES) production function,

$$y = \left[ \sum_{i=1}^{n} (\chi_i l_i)^{1-1/\varepsilon} \right]^{\varepsilon/(\varepsilon-1)}, \tag{1}$$

where $l_i$ denotes the number of workers allocated to the operation of input-variety $i$, $\chi_i$ is the productivity of this variety (i.e. the number of efficiency units embodied in it), $n$ denotes the number of varieties used by the producer, and $\varepsilon \in (1, \infty)$ is the elasticity of substitution across varieties.[19]

Notice that we are implicitly assuming (here and throughout the paper) that the firm uses each input in constant quantities, here normalized to 1. What varies is the number of input varieties and the quantity of labour assigned to each of them—capacity utilization—(which depend on the firm's decisions), and the productivity of each variety (which will be random). In reality, of course, the quantity of each input will also vary, but abstracting from this decision allows us to focus on technological diversification, which comes from an expansion in $n$, without overly complicating the analysis with relatively unimportant issues.

Alternatively one could re-interpret the $n$ varieties not as inputs but as disembodied technologies to turn labour into output. Expanding the number of varieties of such technologies is also likely to both increase productivity and provide technological diversification, so the "disembodied technology" interpretation also serves to potentially explain the empirical relation between volatility and development. Hence, in the rest of the paper we refer to the varieties interchangeably as inputs and as technologies.[20]

Suppose first that the input productivities $\chi$ are non-random, and all normalized to $\chi_i = 1$. Since all technologies are symmetric in this deterministic setup $l_i = l/n$ for all $i$, with $l$ denoting the total number of employees working at the firm. We can then rewrite (1) as

$$y = n^{1/(\varepsilon-1)} l. \tag{2}$$

Labour productivity $(y/l)$ is increasing in the number of varieties, since varieties are imperfect substitutes ($\varepsilon < \infty$). This is the usual "love of variety" effect of many endogenous growth models (Romer 1990, Grossman and Helpman 1991). The effect is stronger the lower is $\varepsilon$, that is, the less substitutable varieties are. Intuitively, if varieties are highly substitutable, any additional variety is less needed. To rule out explosive growth, we assume $\varepsilon \geq 2$.[21]

[19] As usual in endogenous growth models, we assume that $\varepsilon > 1$, that is, technologies are gross substitutes.

[20] Yet another interpretation is that the production function is of the form $y = \left[\sum_{i=1}^{n} {x_i}^{1-1/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)}$, where $x_i$ is the intermediate good produced by the firm by transforming labour through $x_i = \chi_i l_i$. Nothing substantial would change if the inputs were produced by specialized producers and sold to the firm at arm's length; in this case, shocks to $\chi_i$ would translate into input price shocks.

[21] Otherwise the love of variety effect would be so powerful that the aggregate return to varieties would become increasing. Higher levels of development would counterfactually imply increasing rates

Suppose now that variety-specific productivities are independently and identically distributed (non-negative) random variables with constant mean and variance $\sigma^2$. (Later on, we specify a particular distribution function for the shocks to simplify the derivation of the model dynamics). We take the extreme assumption of independence for expositional clarity, but our argument goes through as long as shocks are imperfectly correlated. It is still true that, on average, output per worker will increase in $n$.[22] What is also true is that the variance of $y$ declines in $n$.

The simplest way to see this is by linearising (1) around the mean of each shock:

$$\hat{y} - \hat{l} = \sum_{i=1}^{n} \frac{\mathrm{MP}_i l_i}{y} \hat{\chi}_i = \frac{1}{n} \sum_{i=1}^{n} \hat{\chi}_i, \tag{3}$$

where $\hat{x} \equiv (x - \mathrm{E}\, x)/\, \mathrm{E}\, x$ denotes the infinitesimal deviation of variable $x$ from its mean in proportional terms, and $\mathrm{MP}_i$ denotes the marginal product of technology $i$. The last equality follows from Euler's theorem, and the fact that varieties are *ex ante* symmetric. The proportional variance of firm-level labour productivity is then

$$\mathrm{Var}(\hat{y} - \hat{l}) = \frac{\sigma^2}{n}. \tag{4}$$

The variance is declining in $n$, the number of technologies. This is a simple application of the law of large numbers: the variance of the average of $n$ independent random variables is proportional to $1/n$.

Notice that we did not need to specify a distribution for the productivity shocks, which highlights that technological diversification at the firm level works for a variety of potential shocks. In what follows, and in the interest of analytical simplicity, we specify a particular stochastic process for productivity to characterize the dynamics of technological diversification in a multi-firm economy. As it will become clear, in a multi-firm economy, not only the overall number of varieties in the economy will be relevant for volatility, but also how the usage of those varieties is distributed across firms.

## 3.2 The Dynamic Model: Aggregate economy

As shown in the previous exercise, volatility and labour productivity at the firm level depend crucially on the level of technological complexity of the firm, represented by

of return on capital inputs, inconsistent with observed development patterns (see Caselli and Feyrer (2007)).

[22] To see why, note that $y^{1-1/\varepsilon} = \sum (\chi_i l_i)^{1-1/\varepsilon}$ is the sum of independent random variables with positive support. Adding more of these variables raises the sum in terms of first-order stochastic dominance. Then $E\,(y)$, the mean of a monotonic transformation of $y^{1-1/\varepsilon}$, has to increase.

the number of varieties. In this section we study how the overall level of technological complexity in a multi-firm economy is determined and how it evolves over time, by endogenizing firms' decisions to invest in new varieties. Much as in models of endogenous growth, firm owners will be attracted by greater profit opportunities. To spell out the dynamics of the model, we specify the aggregate and individual-firm setup, and the stochastic properties of the productivity process.

There is a unit mass of monopolistically-competitive firms, indexed by $j$, each producing a differentiated product. The output of the final good is a CES aggregate of firm-level outputs,

$$Y(t) = \left[\int_0^1 y(j,t)^{(\varepsilon-1)/\varepsilon}\,\mathrm{d}j\right]^{\varepsilon/(\varepsilon-1)}, \tag{5}$$

where $y(j,t)$ is the output produced by firm $j$ at time $t$, and $\varepsilon \in (1,\infty)$ is the elasticity of substitution across firms. As described before, each individual firm produces output by combining a variety of inputs through the CES production function

$$y(j,t) = \left[\sum_i [\chi_i(t) l_i(j,t)]^{1-1/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)}, \tag{6}$$

where $\chi_i(t)$ is the productivity of variety $i$ at time $t$, $l_i(j,t)$ is the number of workers allocated to variety $i$ by firm $j$ at time $t$, and $\varepsilon$ is the elasticity of substitution between varieties.[23] For analytical convenience, we assume the elasticity of demand in (5) to be the same as the elasticity of substitution between varieties. This assumption is satisfied naturally if varieties represent different brands valued by the consumer, in which case the elasticity of demand and the elasticity of substitution are equal. As will become clear later, this assumption will ensure that profits be linear in the number of varieties, simplifying the algebra of aggregation. It can, however, be dispensed with at the cost of additional algebra and virtually no gain of insight.

Before specifying the rest of the model in detail we offer a brief informal preview of the main ingredients. Firms add new varieties to the range of inputs they use by engaging in some adoption effort (e.g. to learn how to use it). In particular, they invest some resources in an adoption process, which succeeds the sooner the more resources the firm invests. In deciding how much to invest in adoption each firm seeks to maximize profits, and since firms are risk-neutral technological diversification is not the goal of this process. Hence the adoption part of the model is very similar to standard expanding-variety models, except that the adoption goes on simultaneously

[23] The sum is over all varieties $i$ in use by the firm.

in multiple firms and, due to the random elements of the model, it implies that different firms will have different number of varieties at the same time.

The new feature of the model is that input varieties are subject to productivity shocks. In particular, once a new variety has been added to a firm's range of inputs, it becomes a permanent part of it until a random shock makes it unusable forever (this assumption is motivated below). This "failure shock" is variety-specific, so it hits all firms that happen to be using that particular input. Such firms see their range of varieties shrink by one. The aggregate effects of such shocks depend, therefore, on the distribution of varieties across different firms. Hence to study the evolution of volatility over time it is necessary to keep track of this distribution.

## 3.3 Productivity shocks

Time is continuous. Varieties have a constant productivity (normalized to 1) during their random lifetime, after which they cease to contribute to production. In our notation, $\chi_i(t)$ equals 1 until time $T_i$, when it falls to 0. $T_i$ is the (random) date of failure of this technology. The arrival of failures for a given variety $i$ is common to all firms using this variety, and it follows a Poisson process with arrival rate $\gamma$. Failures are independent across varieties.

Because failure arrives with a Poisson process, the lifetime follows an exponential distribution with parameter $\gamma$; the probability that $T_i \leq t$ is hence

$$\Pr(T_i \leq t) = 1 - e^{-\gamma t}.$$

Clearly, conditional on variety $i$ working at time 0, the distribution of $\chi_i(t)$ is given by

$$\chi_i(t) = \begin{cases} 1 & \text{with prob. } e^{-\gamma t}, \\ 0 & \text{with prob. } 1 - e^{-\gamma t}. \end{cases} \tag{7}$$

Substituting this productivity into the production function,

$$y(j,t) = \left[ \sum_{i:\chi_i(t)=1} l_i(j,t)^{1-1/\varepsilon} \right]^{\varepsilon/(\varepsilon-1)}.$$

We denote the number of productive varieties used by firm $j$ at time $t$ by $n(j,t)$. Given that all productive varieties are identical, firms will allocate the same number of workers to each,

$$l_i(j,t) = \bar{l}(j,t),$$

so that the production function can be written as

$$y(j,t) = \bar{l}(j,t) n(j,t)^{\varepsilon/(\varepsilon-1)}.$$

Note that this formula implicitly assumes that labour can be reallocated at no cost immediately after a shock is realized. This is exclusively done for simplicity; introducing reallocation costs will magnify the loss of negative technology shocks and mitigate the immediate gain from successful adoption, but will not alter the main results.

Our main motivation for the choice of the stochastic process in (7) is analytical tractability. It dramatically simplifies the decision problem of the firm, because there is only one firm-level state variable to keep track of: Since the productivity of each variety can only take the values zero and one, firms only care about the *set* of varieties that are still productive. Moreover, the symmetry of the varieties, together with the memoryless dynamics, ensures that it is only the *number* of productive technologies that matter. Since ours is the first paper to introduce random time variation in the productivity of varieties in an expanding-variety model, we are obviously not aware of alternative approaches that keep the algebra manageable.

While analytical tractability is the main consideration here, (7) does describe a class of relevant input-specific shocks, namely shocks that make an input completely unavailable. This can occur and has occurred in the case of some natural resources that exist in finite quantities. Here the canonical example familiar from history textbooks is the 19th century "guano crisis". Guano was widely used as a fertilizer to increase crop yields during the early 19th century all over the world. In the second half of the century the reserves run out (largely due to the Peruvian government's mismanagement) and the fertilizer became unavailable, causing a major disruption in agriculture - particularly in countries that did not use a more diversified set of fertilizers including nitrates and mined rock phosphate.[24] Today of course the talk is of the depletion of oil reserves, an example no less relevant because it has not (yet) materialized.

An input does not need to be a natural resource to become unavailable. In 1993 an explosion in a Sumitomo plant in Japan led to the annihilation of two thirds of the world supply of the high-grade epoxy resin used to seal most computer chips, causing shortages and price hikes in the semiconductor industry for several months. More generally, disasters of various nature can destroy the output of highly-specialized producers of intermediates. Furthermore government regulation or trade policy can stop the production or import of certain intermediate products. Human capital is not immune from such shocks either: Pol-Pot and Mao Zedong wiped out the human capital of an entire generation in their respective countries.

Even if not taken literally, the process in (7) can also be seen as a short-cut to model less radical disruptions; in that spirit, shocks to $\chi_i$ can result, for example,

[24] The guano crisis was a major shock. President Fillmore's State of the Union adress in 1850 devotes a large section to it.

from increases in the cost of production or import price of a variety (or from the price of an input needed to use that variety, such as the price of oil or other fuels), from trade disruptions, weather shocks that render a variety useless or severely hinder its transportation to its destination, etc. Note that, though the Poisson process may suggest irreversibility, in practice the failure of a given variety in the model needs not be completely irreversible, since a variety can in principle be put back into place, provided that firms repay the adaptation (or adoption) costs, which we later describe.[25]

## 3.4 Static decisions of a firm

The firm hires workers in competitive labour markets; at time $t$ it faces a wage rate $w(t)$ (which depends on the aggregate state of the economy, and is taken as given by individual firms). The only firm-level state variable is the number of technologies in use at time $t$, $n(j,t)$. The marginal cost is given by:

$$\left[\sum_{i=1}^{n(j,t)} w(t)^{1-\varepsilon}\right]^{1/(1-\varepsilon)} = w(t)n(j,t)^{1/(1-\varepsilon)}.$$

Firms using more varieties have lower marginal costs.

Since firms engage in monopolistic competition, each firm faces an iso-elastic demand with elasticity $\varepsilon$: $y(j,t) = Y(t)p(j,t)^{-\varepsilon}$, where aggregate output $Y(t)$ is taken as the numeraire, and $p(j,t)$ is the price charged by firm $j$ at time $t$. Profit maximisation implies that the firm charges a constant $\varepsilon/(\varepsilon-1)$ markup over its marginal cost:

$$p(j,t) = \frac{\varepsilon}{\varepsilon-1} w(t)n(j,t)^{1/(1-\varepsilon)},$$

and its revenues are given by

$$p(j,t)y(j,t) = Y(t)p(j,t)^{1-\varepsilon} = Y(t)\left[\frac{\varepsilon w(t)}{\varepsilon-1}\right]^{1-\varepsilon} n(j,t). \tag{8}$$

Profits are a constant $1/\varepsilon$ fraction of revenues,

$$\pi(j,t) = p(j,t)y(j,t)/\varepsilon = \frac{1}{\varepsilon}Y(t)\left[\frac{\varepsilon w(t)}{\varepsilon-1}\right]^{1-\varepsilon} n(j,t) = A(t)n(j,t),$$

where $A(t) \equiv \frac{1}{\varepsilon}Y(t)\left[\frac{\varepsilon w(t)}{\varepsilon-1}\right]^{1-\varepsilon}$. Since an individual firm takes $Y(t)$ and $w(t)$ as given, $A(t)$ is also given from a firm's perspective.

[25] So, for example, a trade interruption might make a technology or variety temporarily unavailable, but the variety can potentially come back into use after firms reinvest in its adoption or adaptation.

The total wage bill of the firm is a constant fraction, $(1 - 1/\varepsilon)$, of total revenues and hence it is also linear in $n$:

$$w(t)l(j,t) = (\varepsilon - 1)A(t)n(j,t).$$

We can thus express firm-level employment as

$$l(j,t) = \frac{(\varepsilon - 1)A(t)}{w(t)} n(j,t).$$

Firms with twice as many varieties employ twice as many workers. The number of varieties can be therefore interpreted as an index of firm size, whether the latter is measured by employment or by total sales. Note, finally, that the number of workers assigned to an individual variety, $\bar{l}(j,t) = l(j,t)/n(j,t)$ is independent of the number of varieties currently in use by the firm $n(j,t)$, and hence equal for all firms, a result that shall prove useful when aggregating across firms:

$$\bar{l}(t) = \frac{l(j,t)}{n(j,t)} = \frac{(\varepsilon - 1)A(t)}{w(t)}. \tag{9}$$

## 3.5 Technology adoption

As in Romer (1990) and Grossman and Helpman (1991), adopting new varieties is a costly activity.[26] For analytical convenience, we assume that investment in adoption pays off after a random waiting time. Higher investment in adoption results in a shorter expected waiting time for the next variety. Specifically, following Klette and Kortum (2004) we assume that the adoption of a new variety requires both a stock of knowledge (embedded in current technologies, $n$) and a flow of investment. If the firm spends $I(j,t)$ units of the final good to adopt a new variety, the adoption will be successful with a Poisson arrival rate $\Lambda(j,t) = f[I(j,t), n(j,t)]$, where $f(.,.)$ is a standard neoclassical production function subject to constant returns to scale and satisfying the Inada conditions.[27]

Continuing with our example from agriculture, a firm that seeks to adopt, say, a new variety of fertilizer, will need to incur in costly activity, which might include the effort to find the appropriate type and dose for its soil and crop, the build up of some infrastructure to spread it, etc. The more the firm invests and the more productive or

[26] Adoption costs can be also thought as the cost of research and development of new varieties. For developing countries, however, referring to adoption, adaptation, or imitation costs seems more appropriate.

[27] The random, “memoryless” adoption process ensures that we do not have to track past adoption investment flows of the firm. This is a standard simplifying assumption in endogenous growth models.

bigger (and hence more knowledgeable) the firm is, the more likely the variety will be put into place sooner.

Let $\lambda(j,t) = \Lambda(j,t)/n(j,t)$ denote the adoption *intensity* by firm $j$ at time $t$. By the CRS property of $f$, the flow cost of this adoption intensity is

$$I(j,t) = g[\lambda(j,t)]n(j,t),$$

where $g(.)$ is the inverse of $f(.,1)$, an increasing, convex function with $g(0) = g'(0) = 0$, $\lim_{x\to\infty} g'(x) = \infty$.

As mentioned, technological diversification in this model is not driven by risk aversion. To stress this point, we next characterize the optimal rate of technology adoption in the case of risk-neutral agents. In Appendix B we characterize adoption under complete financial autarky and risk-averse investors. We do this to highlight that there is technological diversification in both cases and that the incentive to diversify does not hinge on the financial structure of the economy or the degree of risk aversion (though quantitatively they may affect these incentives).

Identical risk-neutral households maximize the present value of consumption, discounted at the rate $\rho$:

$$\mathcal{U} \equiv \int_{t=0}^{\infty} e^{-\rho t} C(t)\,\mathrm{d}t.$$

The Euler equation pins down the riskless rate in the economy at $r(t) = \rho$. Investors maximize the expected present value of profits, discounted with the rate $\rho$.

To ensure non-negative growth and a finite value for the firm, we make the following parameter restrictions on $\gamma$, $\rho$ and the cost of adoption:

$$g(\gamma) + \rho g'(\gamma) \leq L/2 \quad \text{and} \quad \lim_{x\to\gamma+\rho} g(x) = \infty. \tag{10}$$

The first condition ensures that a variety is profitable enough so that it is worth replacing every failed variety. The second condition ensures that adoption is costly enough so that the growth rate of the economy will never exceed $\rho$, the subjective discount rate.

Let $V[n(j,t),t]$ denote the expected present discounted value of profits for a firm $j$ with $n(j,t)$ varieties at time $t$.

$$V[n(j,t),t] = \mathrm{E}_0 \int_{t=0}^{\infty} e^{-\rho t}[\pi(j,t) - I(j,t)]\,\mathrm{d}t = \\ \mathrm{E}_0 \int_{t=0}^{\infty} e^{-\rho t}\{A(t) - g[\lambda(j,t)]\}n(j,t)\,\mathrm{d}t. \tag{11}$$

Profits accrue from operations, $\pi(j,t)$, netting out the costs of investment, $I(j,t)$. By the homogeneity of the problem, each term is linear in $n(j,t)$. Future profits are discounted by $\rho$. The stochastic dynamics of $n(j,t)$ is described as follows. In each time period of infinitesimal length $h$, one of the inputs fails with probability $\gamma n(j,t)h$ (omitting higher order terms), decreasing $n$ by 1, or the firm becomes successful in adopting a new input (with probability $\lambda(j,t)n(j,t)h$), increasing $n$ by 1.

The Bellman equation describing the decision problem and the value of the firm is

$$\begin{aligned}\rho V[n(j,t),t] = &\max_{\lambda}\Big\{A(t)n(j,t) - g(\lambda)n(j,t) \\ &+ \lambda n(j,t)\{V[n(j,t)+1,t] - V[n(j,t),t]\} \\ &+ \gamma n(j,t)\{V[n(j,t)-1,t] - V[n(j,t),t]\} \\ &+ \lim_{\Delta t\to 0}\mathrm{E}_t\left\{V[n(j,t+\Delta t),t+\Delta t] - V[n(j,t+\Delta t),t]\right\}/\Delta t\Big\}.\end{aligned} \tag{12}$$

The opportunity cost of the value of the firm ($\rho V(n,t)$) equals the sum of (i) flow profits net of adoption costs ($An - g(\lambda)n$), (ii) capital gain from successful adoption of a new technology (which occurs with hazard rate $\lambda n$), (iii) capital loss if any of the $n$ varieties fails (each of which has a hazard rate $\gamma$), and (iv) exogenous capital gains (due to changes in the aggregate environment affecting profitability).

The state of the aggregate economy only affects the firm's adoption problem through its impact on profits per variety, $A(t)$. To characterize the dynamics of $A(t)$, we close the model by considering the conditions for labour market and good market clearing.

## 3.6 General equilibrium

From (9), individual output for firm $j$ is given by:

$$y(j,t) = n(j,t)^{\varepsilon/(\varepsilon-1)}\bar{l}(t),$$

which can be aggregated across firms to obtain

$$Y(t) = \left[\int_0^1 n(j,t)\,\mathrm{d}j\right]^{\varepsilon/(\varepsilon-1)}\bar{l}(t).$$

Let us denote by $N(t)$ the overall number of varieties across all firms: $N(t) = \int_0^1 n(j,t)\,\mathrm{d}j$. Note that the same variety can recur multiple times in $N(t)$; to stress the point—and because it will become relevant later—let us denote by $\tilde{N}(t)$ the number of *different* varieties in the economy at time $t$. Also, because the measure of firms is normalized to unity, $N(t)$ can be thought of as the *average* number of varieties across firms, hence, $N(t) \le \tilde{N}(t)$.

Using the labour market clearing condition $N(t)\bar{l}(t) = L$, where $L$ is a fixed labour supply, we can express aggregate output as

$$Y(t) = N(t)^{1/(\varepsilon-1)}L. \tag{13}$$

Aggregate GDP per worker, $Y/L$, is an increasing function of $N$, the overall number of varieties used in the economy. We can hence think of $N$ as an index of development.

Equation (13) pins down aggregate demand as $Y = N^{1/(\varepsilon-1)}L$. To derive equilibrium wages, note that each firm has a constant profit margin $(1/\varepsilon)$. The total wage bill is a fraction $1 - 1/\varepsilon$ of total output, which pins down the wage rate at

$$w(t) = \left(1 - \frac{1}{\varepsilon}\right) N(t)^{1/(\varepsilon-1)}. \tag{14}$$

Equations (5) and (14) together imply that the demand shifter of the firm is

$$A(t) = \frac{1}{\varepsilon} N(t)^{(2-\varepsilon)/(\varepsilon-1)} L. \tag{15}$$

Profits per variety decrease with aggregate productivity or remain unchanged when $\varepsilon = 2$. There are two opposing forces at play. On the one hand, since varieties are substitutes, higher productivity of competitors implies lower demand for a particular firm's product. On the other hand, since varieties are *imperfect* substitutes, there is a demand externality: more aggregate varieties raise income and hence demand for every firm's product. As long as the elasticity of substitution is not too low, the first effect dominates. When $\varepsilon = 2$, the two effects cancel out.

### 3.7 The Balanced-*Expected*-Growth Path

In a large class of endogenous growth models (that do not feature aggregate uncertainty) a restriction on parameters is needed to insure the existence of a balanced-growth path, or an equilibrium path for the economy where the growth rate of GDP is constant. Our model is no exception: if we assume away productivity shocks to existing varieties, a deterministic balanced-growth path exists if and only if $\varepsilon = 2$.

Our model differs from standard endogenous-growth models in that there are aggregate shocks affecting the productivity of existing varieties. The important consequence of this is that aggregate output growth in every period is itself a random variable: when a lot of varieties are hit (or a variety used by many firms), output growth will be slow; when few varieties are subject to failure (or ones used by only few firms), output will grow fast. However, it turns out that under the same assumption $\varepsilon = 2$ it is possible to characterize the dynamic behaviour of the expected growth rate. In particular, at any time $t$ and for any previous history of the economy the expected (instantaneous) growth rate $E(dY(t)/Y(t))$ is constant. We refer to the resulting equilibrium path as the balanced-expected-growth path.[28]

[28] Note that for $\varepsilon > 2$ there is no long-run growth in the non-stochastic version of the model (absent population growth) and the economy stagnates. Stochasticity adds deviations to the level of output in the stagnated economy, but cannot generate balanced growth.

We next derive the firm-level and aggregate dynamics for the balanced-expected growth path, and then confirm that the balanced-expected-growth path exists and is the only equilibrium of the model. We also compute the expected growth rate on the balanced-expected-growth path

### 3.7.1 Firm-level dynamics

**Proposition 1.** In the balanced-expected-growth path, the optimal adoption *intensity* $\lambda$ is constant (independent of $n(j,t)$ and calendar time $t$). The value of the firm is of the form $V[n(j,t),t] = v \cdot n(j,t)$, where $v$ and $\lambda$ are jointly determined by

$$g'(\lambda) = v, \quad (16)$$

$$L/2 - g(\lambda) = (\rho - \lambda + \gamma)\, v. \quad (17)$$

Adoption intensity, $\lambda$, is positive and unique. It is increasing in market size $L$, decreasing in the discount rate, $\rho$, decreasing in the probability of failure, $\gamma$, and decreasing with an upward shift in adoption costs.

The first equation is the first-order condition for optimal adoption: the marginal cost of adoption, $g'(\lambda)$, should equal the marginal value of an additional variety, $v$. The second equation defines the value of a variety recursively: given the optimal adoption intensity, current profits, $L/2-g$, should compensate for the opportunity cost of capital, $\rho v$, as well as for the expected capital loss, $(\gamma - \lambda)v$. The proof of this and all the remaining propositions are in the appendix.

The linearity of the program ensures that the firm's problem is scale independent. The intensity of adoption, and therefore the firm's growth rate is independent of $n$. We can now fully characterize the dynamics of a firm.

**Proposition 2.** In the balanced-expected-growth path, the expected growth of sales for the firm is $\lambda - \gamma$, and the variance of sales growth is $[\lambda + \gamma]/n(j,t)$.[29]

Equation (8) shows that sales are a linear function of $n(j,t)$, hence their growth rate equals the growth rate of $n(j,t)$. The expected growth in the number of varieties equals the rate of technology adoption minus the rate of technology failure, $\lambda - \gamma$. The variance of sales growth is driven by the two shocks the firm faces: the randomness of the adoption process and variety failures. Hence the variance of an individual variety is $\lambda + \gamma$. Total sales volatility then declines with $n(j,t)$ by the law of large numbers. (The Appendix gives a formal proof.)

[29] We focus on the behavior of sales growth, for which data are available at the firm level. (See Section 2.)

This proposition implies that bigger, more productive firms are less volatile,[30] an empirical regularity we discussed in Section 2.

### 3.7.2 Aggregate dynamics

To understand the dynamics of aggregate GDP, we need to characterize the dynamics of $N(t)$. There are two types of shocks affecting $N(t)$. First, successful adoption of some firms will move them from $n$ varieties to $n+1$ varieties. By Proposition 1, all firms adopt new varieties with intensity $\lambda$, independently of $n$. We assume that firms try to adopt technologies with lower indexes first. A firm of size $n$ has thus access to technologies $1, 2, ..., n$ and would, upon success, adopt technology $n+1$ next. We assume that the success of adoption is completely idiosyncratic, that is, independent across firms. Because there is a continuum of firms, a non-stochastic fraction of them is going to become successful in adoption at any point in time. This means that, in this setup, adoption does not contribute to aggregate uncertainty.

The second type of shock is the failure of a particular technology $k$. This decreases the number of varieties by 1 for all firms that use variety $k$.[31] Because there is a positive mass of these firms, this shock induces an instantaneous *jump* in $N$. The aggregate impact of the shock (and, ultimately, aggregate volatility) will depend on the measure of firms using technology $k$.

By a change of variables, the aggregate number of varieties can be written as $N(t) = \sum_{i=1}^{\tilde{N}(t)} im_i(t)_i$, where $m_i(t)$ is the mass of firms that have exactly $i$ varieties at time $t$. (Recall that $\tilde{N}(t) > 0$ is the number of *different* varieties in the economy, which also evolves endogenously in the model, as firms with the largest number of varieties adopt new ones or technological shocks destroy existing ones.[32]) Given our assumption on the ranking of varieties, the firms that use variety $k$ are exactly the ones that have at least $k$ varieties: $\sum_{i=k}^{\tilde{N}(t)} m_i(t)$.

[30] As is generally the case in monopolistic competition models, $n$ is an index of both productivity and size.

[31] When a variety is shocked, the index of all varieties with higher index are readjusted so as to leave no wholes in the ordering.

[32] By our assumption on the ranking of varieties, $\tilde{N}(t)$ is the upper bound at time $t$ of the support of the distribution of productive varieties $i$; it is also the maximum number of productive varieties used by the biggest (most productive) firm in the economy. $\tilde{N}(t) = \max_j [n(j,t)]$.

Formally, we can describe the dynamics of $N(t)$ as follows:

$$N(t+h)=\begin{cases} N(t)+\lambda N(t)h & \text{with } \Pr = 1-O(h) \\ N(t)+\lambda N(t)h - 1\cdot\sum_{i=1}^{\tilde{N}(t)} m_i(t) & \text{with } \Pr = \gamma h + o(h) \\ \vdots & \vdots \\ N(t)+\lambda N(t)h - 1\cdot\sum_{i=k}^{\tilde{N}(t)} m_i(t) & \text{with } \Pr = \gamma h + o(h) \\ \vdots & \vdots \\ N(t)+\lambda N(t)h - 1\cdot m_{\tilde{N}(t)}(t) & \text{with } \Pr = \gamma h + o(h) \end{cases} \tag{18}$$

Over an infinitesimal $h$ period of time, a non-stochastic $\lambda N$ measure of firms expand their varieties by 1, and with probability $\gamma h$, variety $k$ fails, affecting a measure $\sum_{i=k}^{\tilde{N}(t)} m_i$ of firms.

The aggregate state of the economy is fully characterized by the entire size distribution of firms, $\{m_1(t), m_2(t), ..., m_{\tilde{N}}(t)\}$. Aggregate productivity (and hence per-variety profitability) is a function of the first moment of this distribution, $N(t)=\sum_{i=k}^{\tilde{N}(t)} i m_i(t)$.

With $\varepsilon = 2$, GDP is linear in $N$,

$$Y(t) = N(t)L.$$

The expected growth rate of $Y$ is hence simply the expected growth rate of $N$,

$$\mathrm{E}(\mathrm{d}Y/Y) = (\lambda-\gamma)\,\mathrm{d}t, \tag{19}$$

the speed of adoption minus the speed of "depreciation." We next characterize long-run growth (and therefore $\lambda$) and aggregate volatility.

Long-run growth in the economy can be characterized by its mean, as stated in the following proposition:

**Proposition 3.** A balanced-expected-growth path exists; on the balanced-expected-growth path the expected growth rate $x$, is implicitly defined by

$$[\rho - x]\, g'(\gamma + x) = L/2 - g(\gamma + x), \tag{20}$$

with $x \in [0, \rho)$. The growth rate is increasing in $L$; decreasing in the discount rate, $\rho$; decreasing in the probability of a technology shock, $\gamma$; and decreasing with an upward shift in $g(.)$ (costlier adoption). In equilibrium the economy is always on the balanced-expected-growth path.

## 3.8 Volatility Dynamics along the Balanced-Expected-Growth Path

Since at any time $t$ (instantaneous) GDP growth $dY(t)/Y(t)$ is a random variable, it not only has an expected value but also a variance. Unlike the expected value, however, this variance is not constant, even on the balanced-expected-growth path. (Notice that if it were, the model would have no hope of explaining the cross-sectional patterns of volatility and development that motivate the paper). Instead, it depends on the set of technologies in use, as well as their distribution among firms. In general, these depend on the particular history of shocks that have hit the economy, so the variance must be computed by numerical simulation. Before we turn to this task, we offer some theoretical results that both help with the simulations and provide some intuition on the main mechanism at play.

The volatility of $N$ (and hence the volatility of $Y$) depends on the whole distribution of varieties used by firms. If some varieties are used by more firms than others, then shocks affecting these varieties are going to have a larger impact on GDP. Letting $s_k$ denote the contribution to output of technology $k$, $s_k = \sum_{i=k}^{\tilde{N}(t)} m_i/N$, we can express aggregate variance as

$$\mathrm{Var}(\mathrm{d}N/N) = \gamma \left[ \sum_{k=1}^{\tilde{N}(t)} s_k^2 \right] \mathrm{d}t.$$

Intuitively, a shock hitting variety $k$ reduces $N$ by a fraction $s_k$. Given that this has probability $\gamma\,\mathrm{d}t$, the aggregate variance caused by this shock is $\gamma s_k^2\,\mathrm{d}t$. Because variety-specific shocks are independent, we can simply add up the individual variances. The next proposition states this result formally.

**Proposition 4.** In the balanced-expected-growth path, the variance of GDP growth rates is given by

$$\mathrm{Var}(\mathrm{d}Y/Y) = \gamma\,\mathrm{d}t \sum_{k=1}^{\tilde{N}(t)} s_k^2. \tag{21}$$

To gain some intuition for formula (21), consider some simple examples. If all firms use just one variety, the sum on the right-hand side is one. This leads to the highest possible level of aggregate volatility, $\gamma\,\mathrm{d}t$. If all firms use $\tilde{N}$ different varieties, the contribution of each variety to GDP is $s_k = 1/\tilde{N}$ and the sum equals $1/\tilde{N}$. In this particular case, the sum decreases inversely with the number of varieties and volatility (the standard deviation of GDP growth rates) hence declines at the rate $1/\sqrt{\tilde{N}}$.[33]

[33] Note that in this example, $\tilde{N}$, the number of different varieties, coincides with $N$, the total number of varieties, since the measure of firms with $\tilde{N}$ varieties is 1. Note also, that since firms are

The term $\sum_{k=1}^{\tilde{N}} s_k^2$ can be construed as an index of the technological concentration (or the inverse of an index of technological diversification) of the economy and it is the key determinant of volatility. In a multiple-firm economy, volatility depends not only on the overall number of varieties, $N(t)$ (or the number of different varieties $\tilde{N}(t)$) but on the degree of diversification in the usage of different varieties. As mentioned, both $\tilde{N}(t)$ and the shares $s_k$ are history dependent. For example, there can be histories of shocks such that $\tilde{N}(t)$ is very large but very few firms use input number $\tilde{N}$; histories where $\tilde{N}(t)$ is low, but used by many firms; etc.

Through the introduction of *new* varieties, technological progress increases the degree of technological diversification (and hence lowers volatility) while increasing the level of development. This imparts a natural tendency for a negative correlation between volatility and development that will be prevalent in our numerical analysis.

Note, however, that in principle the relation between volatility and development does not need to be always strictly negative.[34] To understand this, it is convenient to distinguish among the three elements that shape the behaviour of aggregate volatility and development in the model. The first is the adoption of *new* varieties by firms at the technological edge (i.e., firms with the maximum number of varieties, $\tilde{N}$). The second is the adoption of a variety that already exists in the economy by firms that are relatively less sophisticated (i.e., firms with fewer than $\tilde{N}$ varieties). The third is a shock that destroys a variety.

The first element unambiguously leads to an increase in output and a decrease in volatility. The second element always leads to an increase in output, but its effect on

not symmetric ex-post (only a fraction of firms is successful in adoption), this result cannot hold at every point in time.

[34] E.g., a given economy can have a lower overall number of varieties $N(t)$ (that is, a lower level of development) than a second one and at the same time display a higher level of diversification and hence lower volatility. To see this, consider the cross-sectional comparison of two different economies, $A$ and $B$ with identical probabilities of failure $\gamma$. In economy $A$ a measure $x \in (0,1)$ of firms uses variety 1 and a measure $(1-x)$ uses varieties 1, 2, and 3. In our notation, $\{m_1^A, m_2^A, m_3^A\} = \{x, 0, (1-x)\}$; the overall number of varieties in this economy is $N^A = \sum_{i=k}^{3} im_i^A = 3 - 2x$ and its expected variance is $Var^A = \gamma \sum_{k=1}^{3} s_k^2 = \gamma \frac{1+2(1-x)^2}{(3-2x)^2}$. In economy $B$, all firms use varieties 1 and 2: $\{m_1^B, m_2^B, m_3^B\} = \{0, 1, 0\}$; the overall number of varieties in $B$ is hence $N^B = \sum_{i=k} im_i^B = 2$ and its variance is $Var^B = \gamma \sum_{k=1} s_k^2 = \gamma \frac{1}{2}$. It can be easily shown that $N^A < N^B$ and $Var^A < Var^B$ for $\frac{1}{2} < x < \frac{3}{4}$. The condition $x > \frac{1}{2}$ ensures that there is a large enough fraction of firms in $A$ using only one technology so that the overall number of varieties in economy $B$ is bigger than in $A$; the condition $x < \frac{3}{4}$ ensures that there is a sufficient fraction of firms employing three different technologies in economy $A$ and thus leading to higher diversification of aggregate risk and lower volatility in $A$ relative to $B$. By continuity, the result will also hold for some range of $x$ if some firms in $A$ also use technology 3, that is, when the number of *different* varieties $\tilde{N} = 3$ in use is the same for both economies.

volatility is ambiguous: Though in general it causes a decline in volatility, it might, under certain conditions cause an increase. The third element always causes a decrease in output, while its effect on volatility is also in principle ambiguous: In general it increases volatility, but it might, in some circumstances cause a decline.

To see these mechanisms more clearly, consider them one at a time (that is, shutting down the other two).

1. Adoption of *new* varieties (by firms with $n(j,t) = \tilde{N}(t)$)

Consider first what happens in the model when an infinitesimal number of firms at the technological edge adopt a new variety. Denote the measure of firms by $\Delta$ and use ($'$) to denote the new values of the different variables (after adoption). As these firms have added one more variety, the overall number of varieties in the economy goes up by $\Delta$:

$$N' = N + \Delta.$$

Or, in terms of derivatives, $\lim_{\Delta\to 0} \frac{N'-N}{\Delta} = 1 > 0$. Hence, output unambiguously increases. (Note also that the number of *different* varieties in the economy goes up by 1: $\tilde{N}' = \tilde{N} + 1$.) Before the introduction of the new variety (indexed $\tilde{N}+1$), its contribution to output was $s_{\tilde{N}+1} = 0$. With the introduction of the new variety, the contributions of the different technologies become

$$\begin{aligned} s'_{\tilde{N}+1} &= \Delta/N' \\ s'_i &= N s_i/N' \text{ for all } i \neq \tilde{N}+1 \end{aligned}$$

and the new variance is given by:

$$\mathrm{Var}' = \left(\frac{N}{N'}\right)^2 \left(\sum\nolimits_{i=1}^{\tilde{N}} s_i^2 + (\Delta/N)^2\right)$$

The change in variance is hence $\mathrm{Var}' - \mathrm{Var} = \left(\frac{N}{N+\Delta}\right)^2 \left(\sum_{i=1}^{\tilde{N}} s_i^2 + (\Delta/N)^2\right) - \sum_{i=1}^{\tilde{N}} s_i^2$, or, in terms of derivatives:

$$\lim_{\Delta\to 0} \frac{\mathrm{Var}' - \mathrm{Var}}{\Delta} = -\frac{2}{N} \sum\nolimits_{i=1}^{\tilde{N}} s_i^2 < 0,$$

which is strictly negative.[35] Thus, the introduction of a new variety always increases development and decreases volatility

[35] To see this, note that $\lim_{\Delta\to 0} \frac{\mathrm{Var}'-\mathrm{Var}}{\Delta} = \lim_{\Delta\to 0} \frac{\left(\frac{N}{N+\Delta}\right)^2 - 1}{\Delta} \sum_{i=1}^{\tilde{N}} s_i^2 + \lim_{\Delta\to 0} \left(\frac{N}{N+\Delta}\right)^2 \frac{(\Delta/N)^2}{\Delta}$. The first terms converges to 0/0. By l'Hôpital's rule, it becomes $-\frac{2}{N}\sum_{i=1} s_i^2$. The second term converges to 0.

2. Adoption of existing varieties (by firms with $n(j,t) < \tilde{N}(t)$)

Consider now what happens when an infinitesimal number of firms $\Delta$ adopts a variety $k$, already in use in the economy. (We continue to denote by $(')$ the new values.) As these firms have added one more variety, the overall number of varieties again goes up by $\Delta$:

$$N' = N + \Delta.$$

That is, $\lim_{\Delta\to 0} \frac{N'-N}{\Delta} = 1 > 0$. Hence, output always increases (though the number of *different* varieties in the economy is unchanged: $\tilde{N}' = \tilde{N}$.) Before the adoption of variety $k$, its contribution to output was $s_k > 0$. After adoption, the contribution of the various technologies become

$$\begin{aligned} s'_k &= (Ns_k + \Delta)/N' \\ s'_i &= Ns_i/N' \text{ for all } i \neq k, \end{aligned}$$

and the new variance is given by:

$$\text{Var}' = \left(\frac{N}{N'}\right)^2 \left(\sum_{i=1}^{\tilde{N}} s_i^2 + (\Delta/N)^2 + 2s_k\Delta/N\right)$$

The change in variance is hence $\text{Var}' - \text{Var} = \left(\frac{N}{N+\Delta}\right)^2 \left(\sum_{i=1}^{\tilde{N}} s_i^2 + (\Delta/N)^2 + 2s_k\Delta/N\right) - \sum_{i=1}^{\tilde{N}} s_i^2$, or, in terms of derivatives:

$$\lim_{\Delta\to 0} \frac{\text{Var}' - \text{Var}}{\Delta} = -\frac{2}{N}\left(\sum_{i=1}^{\tilde{N}} s_i^2 - s_k\right),$$

which is negative if and only if $\sum_{i=1}^{\tilde{N}} s_i^2 > s_k$.[36] Hence, as long as $\sum_{i=1}^{\tilde{N}} s_i^2 > s_k$, volatility decreases with the adoption of variety $k$. Intuitively, as long as a variety is not widely used in the economy, increasing its usage provides diversification benefits against other variety-specific shocks and hence lowers aggregate volatility. In contrast, when a variety is already intensely used, increasing its usage makes the economy more exposed to shocks affecting that variety.[37] Note that the firms adopting technology $k$

[36] The limit results from l'Hôpital's rule.

[37] To see this, consider the following numerical illustration. The distribution of the number of firms with exactly $i$ varieties $(m_i)$ is given by $\{m_1, m_2, m_3, m_4\} = \{\frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{4}{10}\}$; the overall number of varieties in the economy is then $N = \sum_{i=k}^{4} im_i = 3.1$ and the shares of each variety in the economy are $\{s_1; s_2; s_3; s_4\} = \{0.32; 0.29; 0.26; 0.13\}$, with $\sum_{k=1}^{4} s_k^2 = 0.272$ (and $Var = 0.272\gamma$). Hence $\sum_{k=1}^{4} s_k^2 < s_2$. Adoption of variety 2 by firms with only variety 1 can hence lead to an increase in output and an increase in volatility. Indeed, if all $m_1$ firms adopt variety 2, we have: $\{m'_1, m'_2, m'_3, m'_4\} = \{0, \frac{2}{10}, \frac{4}{10}, \frac{4}{10}\}$, which implies $N' = 3.2$ and $\{s'_1; s'_2; s'_3; s'_4\} = \{0.31; 0.31; 0.25; 0.13\}$ , leading to $Var' = \gamma \sum_{k=1}^{4} s_k^2 = 0.273\gamma > Var$. Because variety 2 was already widely used, increasing its usage by firms of size 1 made the economy more exposed to shocks to that variety.

always become less volatile, even if aggregate volatility increases (that is, even if the share of that variety in the economy $s_k$ is already big).

3. Negative shock to a variety

Finally, consider the consequences of a negative shock that destroys variety $k$. The number of firms using this variety is $\sum_{i=k}^{\tilde{N}} m_i = Ns_k$ (by definition of $s_k$). The overall number of varieties falls to $N' = (1-s_k)N$ and output correspondingly falls to $Y' = (1-s_k)Y$. (The number of *different* varieties also falls by 1: $\tilde{N}' = \tilde{N} - 1$.) The new shares in the economy are given by:[38]

$$\begin{aligned} s_i' &= s_i/(1-s_k) \text{ for all } i<k, \\ s_i' &= s_{i+1}/(1-s_k) \text{ for all } k \leq i < \tilde{N} \\ s_{\tilde{N}}' &= 0 \end{aligned}$$

and the new variance is given by:

$$\text{Var}' = \frac{1}{(1-s_k)^2}\left(\sum\nolimits_{i=1}^{\tilde{N}} s_i^2 - s_k^2\right).$$

The change in variance is hence:

$$\begin{aligned} \text{Var}' - \text{Var} &= \left[\frac{1}{(1-s_k)^2} - 1\right]\sum\nolimits_{i=1}^{\tilde{N}} s_i^2 - \frac{s_k^2}{(1-s_k)^2} \\ &= \frac{s_k}{(1-s_k)^2}\left[(2-s_k)\sum\nolimits_{i=1}^{\tilde{N}} s_i^2 - s_k\right], \end{aligned}$$

which is positive if an only if $\sum_{i=1}^{\tilde{N}} s_i^2 > \frac{s_k}{2-s_k}$. In words, as long as $s_k$ is not too big, expected volatility increases with the destruction of variety $k$. This happens together with the unambiguous decline in output caused by the destruction of that variety. Volatility might decrease only if the production process relies strongly on variety $k$. In that case, the disappearance of that variety leads to higher diversification for the economy.

Note that because $s_k > \frac{s_k}{2-s_k}$, the destruction of a variety is less likely to induce a positive correlation between volatility and development than the adoption of existing varieties.[39]

[38] Recall that when a shock takes place, we re-index all varieties so as to leave no holes in the ordering. E.g. if an economy has varieties $k = 1,2,3,4$ and variety 3 fails, then, in the following (infinitesimal) period variety 4 is indexed 3 and $k' = 1,2,3$.

[39] Destruction induces a positive correlation when $\sum_{i=1}^{\tilde{N}} s_i^2 < \frac{s_k}{2-s_k}$ whereas the adoption of existing varieties does it when $\sum_{i=1}^{\tilde{N}} s_i^2 < s_k$.

While one can construct examples where the negative relation between volatility and development breaks, in general, the model dynamics generates a negative correlation. This is because the growth process through the steady introduction of new varieties leads on average to both higher levels of development and higher degrees of technological diversification (and hence lower volatility). Only occasionally the adoption of a variety that is already widely used in the economy, or (though less likely) its destruction, can generate a positive correlation between volatility and development. In general, the upward trend in GDP and the downward trend in volatility imparted by the growth process will prevail, yielding a negative correlation, as will become evident in the numerical example. In the long run, as per capita GDP grows without bound, volatility approaches zero, as stated by the following proposition.

**Proposition 5.** As per capita GDP increases without bound, volatility tends to zero.

The intuition is straightforward: Long-run growth of GDP per capita can only be achieved by the addition of new varieties, which reduces volatility, as we have shown above. (The formal proof is in the Appendix.)

This provides a powerful characterization of the long run equilibrium in the economy. As time progresses, the economy will resemble the deterministic economy of standard growth models with expanding varieties. Eventually, volatility will vanish and the economy will reach a stable, non-stochastic growth rate of $\lambda - \gamma$. This highlights that the decline in volatility is a necessary by-product of the development process.

Finally, we should perhaps stress that aggregate volatility depends on the distribution of varieties in the economy, but not on the distribution of firms per se. This is because technology shocks are variety specific but not firm specific.[40]

We next provide a numerical analysis of the model and its predictions.

[40] Note, however, that the distribution of firms can have an impact on the distribution of varieties, as large firms tend to use more varieties, that is, aggregate volatility can change if the relative importance of large and small firms changes. For example, suppose that a fraction $\alpha$ of firms uses two varieties, while the other $1-\alpha$ fraction uses only one. The overall number of varieties is then

$$N = \alpha 2 + (1-\alpha)1 = 1 + \alpha,$$

which is clearly increasing in $\alpha$. The higher the fraction of large firms, the more overall varieties are used.

The variance is then

$$\text{Var} = \gamma \frac{1+\alpha^2}{(1+\alpha)^2},$$

which is decreasing in $\alpha$. Since small firms are using only one variety while large firms use two, reducing the number of small firms and increasing the number of large firms leads to diversification *across varieties.*

# 4 The relation between volatility, growth, and development: A quantitative assessment

Our analysis so far has shown that volatility declines monotonically with the degree of technological diversification (as defined, inversely, in equation (21)) and that, ceteris paribus, the introduction of a new variety in the economy increases the level of development and the degree of technological diversification, thus lowering volatility. We have also argued that the growth process, through the expansion in the number of varieties, tends to impart a negative correlation between volatility and development, though this tendency may be overturned under certain histories of shocks; specifically, it is conceivable that countries that use very intensely a few varieties display both a relatively high level of development and high volatility due to their lack of diversification. To establish whether these occurrences are frequent or rare one has to simulate the model.

Our strategy is to generate artificial data by simulating the model for 150 economies (countries) for $T$ periods (we describe how we choose $T$ below). All economies start from the same initial conditions and are governed by the same parameter values. However, shocks are country-specific and different realizations of shocks lead to potentially different equilibrium paths and thus different levels of volatility and development at any point in time. We then analyse the relation between volatility and the level of development for the simulated economies. We also compare patterns of volatility and development in the last 40 years of our simulations to the corresponding patterns in the cross-sectional data from roughly 150 countries that we already examined in Section 2. Note that because the volatility of aggregate GDP depends on the distribution of technologies across firms, our simulations need to keep track of the entire distribution of technology usage across firms at all points in time.

## 4.1 Parametrization and Computation

We focus the analysis on the balanced expected-growth path, and compute a discrete-time approximation of our continuous-time model. In calibrating our discrete-time approximation, a period is interpreted as a quarter. We need to set values for $\gamma$, the arrival rate of technology failures, and $\rho$, the rate of time preference. In principle, we also need to specify and parametrize the cost of adoption function $g(.)$ and the size of the economy, $L$. However, note that $g$ and $L$ only serve to pin down the search intensity $\lambda$, which of course is constant in the balanced-expected-growth path. Hence we let $g$ and $L$ unspecified and calibrate $\lambda$ directly. Finally, we need to specify the number of periods for the simulations, $T$.

The model key parameter, $\gamma$, is in principle difficult to calibrate without directly observing technology shocks. Our strategy is thus to simulate the model for a large grid of values for $\gamma$, ranging from 0.01 to 0.20. Values below 0.01 imply virtually no aggregate volatility; values above 0.20 are unlikely, as they would imply that technologies last on average less than 5 years.[41] Recalling that in our model the expected growth rate is $\lambda - \gamma$, we can then set $\lambda$ so that the annual growth rate is 0.02 (i.e. $\lambda = \gamma + 0.02$). The rate of time preference $\rho$ is set at an annual rate of 4 percent.[42]

We start out with 150 identical economies populated by identical firms of size $n_0 = 5$ and simulate the model in each economy for $T$ periods. Varying $n_0$ is almost equivalent to varying the number of periods $T$ over which the model is run, since with positive expected growth, a longer period (like a higher $n_0$) implies that countries will exhibit on average a higher level of technological sophistication by the end of the period. In the interest of space, we only report the results for different $T$.[43]

In every period, a fraction of firms is successful at adoption and becomes bigger, increasing its number of technologies by 1. In case there is a shock to technology $k$, though, all firms using that technology (that is, all firms of size $k$ and above) shrink by 1. To simulate the model, we resort to discrete-time methods. (Note that the state

[41] Technologies with such short duration exist in practice, but are unlikely to be the norm. Perhaps a way to extend the model is to allow for heterogeneity in the probability of failure across technologies. For now, we concentrate on what we think are the first-order insights of the model by assuming a constant $\gamma$.

[42] To see what these choices might imply for the cost of adoption function $g$ and the size of the economy $L$ consider the following parameterization:

$$g(\lambda) = a(b - \lambda)^{-\phi},$$

where $a > 0$ relates to the overall level of adoption costs, $b > 0$ represents an upper bound on adoption investment, and $\phi > 0$ governs the cost elasticity with respect to adoption. Given this choice of $g$, equations (16) and (17) imply that the choice of $L$ is simply a normalization. If we set the upper bound $b$ to be the sum of our choices for $\rho$ and $\gamma$, and normalize $L$ to $L = 2$, equation (20) simplifies to

$$1 - a(\rho + \gamma - \lambda)^{-\phi} = a\phi(\rho + \gamma - \lambda)^{-\phi}. \tag{22}$$

With an expected growth rate $(\lambda - \gamma)$ equal to 2 percent, and $\rho = 0.04$, this can be rewritten as:

$$\frac{1}{\phi + 1} = a(0.04 - 0.02)^{-\phi}$$

Hence, there are infinitely many combinations of $\phi$ and $a$ that are consistent with our calibration and satisfy assumption (10). For example, $\phi = 4$ implies that the cost of adoption of a new variety, $1/(\phi + 1)$, is 10 percent of total firm sales; this choice pins down the value of the level parameter $a$ at $3.2 \cdot 10^{-8}$. A value of $\phi = 15$ implies a cost of 3 percent of firm sales and $a = 2.05 \cdot 10^{-27}$, and so on. Incidentally, estimates of cost-of-adoption functions are hard to find except for some narrowly defined sectors.

[43] Results on the comparative statics for $n_0$ are available from the authors.

space is already discrete.) We approximate the continuous-time adoption and failure processes as follows. Over a period $\Delta t$, a firm of size $n_t$ adopts $q_{t,\Delta t}$ new varieties, where $q_{t,\Delta t}$ has a Poisson distribution with expected value $\lambda n_t \Delta t$. If there are $N_t$ technologies overall, the number of failures, $k_{t,\Delta t}$, is also Poisson with expected value $\gamma N_t \Delta t$. Once the number of failures is determined, they are randomly allocated across technologies, so that each technology has the same $k_{t,\Delta t}/N_t$ chance of failing. In symbols:

$$q_{t,\Delta t} \sim \text{Poisson}(\lambda n_t \Delta t)$$
$$k_{t,\Delta t} \sim \text{Poisson}(\gamma N_t \Delta t)$$

As $\Delta t$ tends to zero, these processes converge to the continuous-time processes described in Section 3. We take $\Delta t$ to be a quarter of a year. (Since both $\lambda$ and $\gamma$ are fairly small, $\Delta t$ does not need to be too small for the above approximation to be accurate.)

At any point in time, we can take a snapshot of the economy by counting the number of firms in each size bin, $m_{1t}, m_{2t}, ..., m_{\tilde{N}t}$. GDP at time $t$ can then be calculated as $Y_t = L \sum_{i=1}^{\tilde{N}(t)} i m_{it}$. To construct statistics that have the same interpretation as those in our empirical analysis of Section 2 we compute decade averages of (the log of) GDP and (logs of) standard deviations of GDP growth over a decade around the decade mean (our measure of volatility). The only difference with the corresponding statistics in the data are that in the model the raw data are quarterly (for a better approximation of the Poisson process) rather than annual.

Because both adoption and technology shocks are permanent, economies will diverge (or at least will not converge) over time. At any point in time, the only difference between the simulated countries is that they have experienced different histories of shocks. Some got lucky, grew fast, and became stable, others were unlucky, grew slowly (maybe even shrank) and remained volatile. When comparing our simulated world to the data, we treat real-world economies as separate realizations of $T$ years of technology shocks.

Recall that in Section 2 we run a regression of country-level volatility on income for the four decades between 1960 and 2000. To run a similar regression on our simulated panel of countries we need at least 40 years of data. The relevant processes, however, have presumably started well before 1960. Furthermore, with only 40 years of data the model economy experiences too few shocks for the income distribution to meaningfully fan out. Hence, we consider that the length for the simulations should be significantly longer. We report results for a variety of lengths, with a minimum of 100 years, and a maximum of 300 years. For each of these simulations lengths, we report statistics for the last four decades. We stop at 300 years because this is roughly what it takes for the model to reach a point where the negative relationship between volatility and development matches quantitatively the one in the data.[44,45]

## 4.2 Results

We present the main results in Tables 3 through 5. Table 3 shows the model-generated slope coefficients from OLS regressions of decade volatility on average (log) GDP of the decade, pooling data from all of the 150 simulated countries and—for each $T$—the last four decades.[46] This is done for different values of $\gamma$ and $T$. The empirical counterpart is the first column of Table 1.

The Table shows that for most parameter values, the regression coefficients are negative and significant at standard confidence levels and quantitatively comparable to those in the data ($-0.23$). (We come back to this point later.) The only exception is for $\gamma \geqslant 0.15$ and $T = 100$; to understand why the correlation can be positive, recall from Section 3.8 that both the destruction and adoption of intensely used varieties can generate a positive correlation between GDP and volatility. Destruction of a sufficiently highly used variety causes a fall in GDP and an increase in the diversification level of remaining varieties, and hence a decrease in volatility. Adoption of such variety increases GDP and decreases the degree of diversification, thus increasing volatility. At high levels of $\gamma$, it is more likely that each individual variety be hit by a shock; furthermore, a large $\gamma$ also implies a large value for $\lambda$, since the growth rate $(\lambda - \gamma)$ is constant and hence the adoption of any given variety is also more likely. If in addition, an economy is still at very low levels of development and concentrated on very few varieties, it is hence more likely that the resulting correlation be positive. In sum, for a given $T$, a positive correlation is more likely to happen for higher values of $\gamma$ (as the intensity of adoption and the probability of any individual-technology shock is higher); for a given $\gamma$, it is more likely to happen at earlier stages of development (or low levels

[44] More specifically, the model takes 250 years to quantitatively match the empirical relation between volatility and development under $0.05 \leq \gamma \leq 0.10$.

[45] This is similar in spirit to Acemoglu and Zilibotti (1997), who ask how many years it takes for their model to achieve full diversification. In our model, unconditional diversification can never be achieved because the number of different varieties $\tilde{N}(t)$ grows without bounds. Since our goal is to explain the empirical relation between volatility and development, we accordingly ask how many years it takes to generate patterns comparable to the data.

[46] One may be concerned that the regression coefficient is sensitive to the particular realization of shocks these 150 hypothetical economies have experienced. The reported standard errors quantify the uncertainty about this parameter, as estimated by OLS in the simulated sample. We also ran a Monte Carlo study to determine the empirical distribution of the slope parameter from 10,000 repetitions. This confirmed that the distribution is well approximated by a normal distribution with the OLS mean and standard error, hence these two numbers accurately characterize the relationship in the simulated data.

of $T$), since economies will have fewer varieties and will hence be more exposed to shocks.

We also computed the within-country slopes by running similar regressions after controlling for country-specific effects. We do not report these results to economize on space.[47] As in the data, however, the time-series slopes generated by the model tend to be larger in magnitude than the corresponding cross-sectional slopes. The average slope across all $T$ and $\gamma$ is $-0.43$, and is significant at the 1 percent level.

Table 3. Volatility and Development: Slope Coefficients for Different $\gamma$ and $T$

| Number of Years T | Poisson Parameter | | | | | |
|---|---|---|---|---|---|---|
| | 0.01 | 0.02 | 0.05 | 0.10 | 0.15 | 0.20 |
| 100 | **-0.2548** | **-0.4171** | **-0.2627** | **-0.1984** | -0.0132 | **0.2042** |
| | (0.097) | (0.045) | (0.027) | (0.023) | (0.027) | (0.034) |
| 150 | **-0.2936** | **-0.3414** | **-0.2252** | **-0.0861** | **-0.1012** | **-0.1139** |
| | (0.035) | (0.042) | (0.026) | (0.027) | (0.025) | (0.025) |
| 200 | **-0.4091** | **-0.2185** | **-0.2082** | **-0.1966** | **-0.1102** | **-0.0980** |
| | (0.024) | (0.038) | (0.025) | (0.022) | (0.020) | (0.021) |
| 250 | **-0.4086** | **-0.2940** | **-0.2065** | **-0.1953** | **-0.1493** | **-0.1242** |
| | (0.022) | (0.037) | (0.023) | (0.021) | (0.019) | (0.018) |
| 300 | **-0.4141** | **-0.2961** | **-0.1366** | **-0.1284** | **-0.1427** | **-0.0878** |
| | (0.021) | (0.036) | (0.023) | (0.020) | (0.017) | (0.017) |

The Table shows the slope coefficients and standard deviations (in parentheses) from regressions of (log) volatility of annualized quarterly growth rates over a decade on average (log) level of development in the decade for model-simulated data, under different parameter values; a constant (not reported) is included in each regression. The number of periods T indicates the number of years during which the model was run. The regressions use data from the four decades previous to year T. Significance at 1 percent level is highlighted in bold. The number of observations in each regression is 600, corresponding to 150 countries observed in 4 consecutive decades. (Results are not altered by introducing decade-fixed effects.)

Table 3 shows that, for a large grid of parameter values, the slopes generated by the model are quantitatively similar to those observed in the data. This is not enough, however, to show that the model can explain the decline in volatility with development seen in the data, which is the starting motivation of this paper. For this assessment it is important to know not only the slope coefficients but also the degree of dispersion in GDP generated by the model. In the data, in 1960 the standard deviation of log

[47]The results are available from the authors.

GDP was 0.92, and the interquartile range of log GDP was 1.34.[48] Table 4 displays the model-generated standard deviation of (log) GDP and interquartile ranges of (log) GDP at the beginning of the last four decades of simulations for different parameter values. The dispersion generated by the model is in general smaller than that in the data, though as $\gamma$ reaches 0.10 or higher, the model gets reasonably close to the data. Because the model has no mechanism to generate convergence, over time, GDP dispersion tends to increase or remain constant, as appears to be the case in the data.[49]

Table 4. Volatility and Development: Statistics for Different $\gamma$ and $T$

| Statistics | Number of Years T | Poisson Parameter 0.01 | 0.02 | 0.05 | 0.10 | 0.15 | 0.20 |
|---|---|---|---|---|---|---|---|
| Standard Deviation of (Log) GDP | 100 | 0.271 | 0.342 | 0.588 | 0.766 | 0.696 | 0.568 |
| | 150 | 0.308 | 0.378 | 0.642 | 0.663 | 0.736 | 0.635 |
| | 200 | 0.318 | 0.380 | 0.659 | 0.776 | 0.882 | 0.795 |
| | 250 | 0.314 | 0.391 | 0.712 | 0.858 | 0.990 | 0.916 |
| | 300 | 0.314 | 0.393 | 0.734 | 0.918 | 1.076 | 1.022 |
| Interquartile Range of (Log) GDP | 100 | 0.350 | 0.391 | 0.717 | 1.019 | 1.051 | 0.881 |
| | 150 | 0.412 | 0.513 | 0.739 | 0.998 | 1.162 | 0.892 |
| | 200 | 0.433 | 0.496 | 0.777 | 1.147 | 1.364 | 1.121 |
| | 250 | 0.422 | 0.486 | 0.826 | 1.210 | 1.266 | 1.235 |
| | 300 | 0.418 | 0.474 | 0.773 | 1.308 | 1.377 | 1.442 |

The Table displays decadal statistics at the beginning of the interval of analysis (which comprises the four decades previous to year T). The number of years T indicates the number of years during which the model was run.

An appealing way to measure the statistical variation of volatility with the level of development in the data is given by $\hat{\beta} \cdot \sigma_{GDP}$, where $\hat{\beta}$ is the slope regression coefficient reported in Table 1 (equal to -0.23) and $\sigma_{GDP}$ is the standard deviation of log GDP (equal to 0.92).[50] We can then ask what fraction of the statistical variation in the data can be generated by the model, by computing $\frac{\hat{\beta}(T,\gamma)\cdot\sigma_{GDP}(T,\gamma)}{\hat{\beta}\cdot\sigma_{GDP}}$, where $\hat{\beta}(T,\gamma)$ and $\sigma_{GDP}(T,\gamma)$ are the model-generated slope coefficient and the standard deviation of (log) GDP, for different values of $T$ and $\gamma$, reported in Tables 3 and 4, respectively.

We report the ratios in Table 5. The model reaches its maximum explanatory power for $\gamma = 0.10$ and $T = 250$. A value of $\gamma = 0.10$ means that each individual technology

[48] The corresponding values in 2000 were 1.09 and 1.82. The data come from the World Bank's *World Development Indicators.*

[49] The standard deviation of log GDP increased from 0.92 to 1.09 and the interquartile range from 1.34 to 1.82 in the period 1960-2000.

[50] Alternatively, one could use the interquartile range (among other measures of dispersion). The final results look quite similar and are available from the authors.

has a 10-year average lifetime.[51] At this value of $\gamma$, the model needs 250 years to reproduce the empirical patterns.[52] For other sets of parameter values, the model still performs quite well in generating a decline in volatility with the level of development comparable to that in the data.

The simulation exercise leads us to conclude that the technological-diversification model, though stylised, can potentially account for a substantial part of the decline in volatility with development observed in the data. It thus highlights a hitherto neglected feature of expanding-variety models, which makes them suitable to explain the secular decline in volatility and its relation with development.

Table 5. Volatility and Development: Model-Generated Variation for Different $\gamma$ and $T$

| Statistics | Number of Years T | Poisson Parameter | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.01 | 0.02 | 0.05 | 0.10 | 0.15 | 0.20 |
| Fraction of empirical variation generated by the model | 100 | 33% | 67% | 73% | 72% | 4% | - |
| | 150 | 43% | 61% | 68% | 27% | 35% | 34% |
| | 200 | 61% | 39% | 65% | 72% | 46% | 37% |
| | 250 | 61% | 54% | 69% | 79% | 70% | 54% |
| | 300 | 61% | 55% | 47% | 56% | 73% | 42% |

The Table displays the fraction of the statistical variation of volatility with respect to development in the data that can be generated by the model. See text for explanations. Values are computed at the beginning of the interval of analysis (which comprises the four decades previous to year T). The number of years T indicates the number of years during which the model was run.

## 5 Concluding remarks

We argue that technological diversification offers a promising (yet so far overlooked) explanation for the negative relation between volatility and development. We do so by proposing a model in which the production process makes use of different varieties subject to imperfectly correlated shocks. Technological progress takes place as an expansion in the number of input varieties, increasing productivity. The new insight in the model is that the expansion in varieties can also lead to lower volatility of

[51] Incidentally, in the model, the (stochastic) annual depreciation rate of the economy is also $\gamma$, so it is reassuring that $\gamma = 0.10$ is the value that fits best.

[52] Strictly speaking, this means that the technological diversification channel alone should have started around the First Industrial Revolution for it to generate this correlation. In practice of course, other channels may have delayed or anticipated the process.

production. First, as each individual variety matters less and less in production, the contribution of variety-specific fluctuations to overall volatility declines. Second, each additional variety provides a new adjustment margin in response to external shocks, potentially making productivity less volatile. In the model, the number of varieties evolves endogenously in response to profit incentives and the decrease in volatility comes out as a likely by-product of firms' incentives to increase profits. We simulate the model for a large grid of parameter values and find that, for most parametrizations, the model can explain a substantial fraction of the statistical variation in volatility with respect to development observed in the data.

# Appendix

# A Proofs

## A.1 Proof of Proposition 1

Since profits are linear in $n$ and profitability $A$ is independent of calendar time $t$, guess that the form of value function is $V(n,t) = vn$, where $v$ is independent of calendar time.

The first-order condition for optimal investment is equation (16). Equation (17), in turn, results from substituting the guess function into the Bellman equation. We now show that there is a unique pair of $v$ and $\lambda$ satisfying equations (16) and (17), hence the guess function is indeed the solution.

Substituting (16) into (17),

$$(\rho+\gamma)\, g'(\lambda) = L/2 - g(\lambda) + \lambda g'(\lambda).$$

Both sides are continuously differentiable with respect to $\lambda$. At $\lambda=\gamma$, the left-hand side (LHS) is (weakly) less than the right-hand side (RHS) by our assumption on the cost function. Similarly, at $\lambda=\gamma+\rho$, the LHS is greater than the RHS. In between, the LHS is growing faster (declining slower) than the RHS, so there is a unique $\lambda \in [\gamma, \gamma+\rho)$ solving the equation.

The comparative statics can be shown as follows. We totally differentiate the first-order condition with respect to $\rho$, $\gamma$, $\lambda$ and $L$:

$$(\mathrm{d}\rho+\mathrm{d}\gamma)g' + (\rho+\gamma)g''\,\mathrm{d}\lambda = \mathrm{d}L/2 + \lambda g''\,\mathrm{d}\lambda,$$

rearranging,

$$\mathrm{d}\lambda = \frac{\mathrm{d}L/2 - (\mathrm{d}\rho+\mathrm{d}\gamma)g'}{(\rho-\lambda+\gamma)g''}.$$

Because of the convexity of $g$ and the condition that growth does not exceed the discount rate, the denominator is positive. Optimal adoption intensity is then

1. increasing with market size, $L$. Intuitively, if the economy has greater profitability per variety today and greater expected profitability in future periods, then the present discounted value of profits per variety increases, raising the benefits to adoption.

2. decreasing in the discount rate $\rho$. The firm always makes positive profits. Discounting positive profits at a higher rate obviously makes the value of the firm lower, and hence adoption less attractive.

3. decreasing in the failure rate $\gamma$. The failure rate acts as a depreciation rate, introducing additional discounting into the present value problem. Higher discounting reduces the incentive to invest in new varieties.

4. decreasing with an upward shift in adoption costs, $g$. Suppose adoption costs are $Bg()$, where $B > 1$. The first-order condition is homothetic in $g$ and $L$, so the solution under the new adoption cost is the same as under the old adoption cost but a small market size, $L/B/2$. By (i), this leads to lower adoption intensity.

## A.2 Proof of Proposition 2

The proof follows directly from the following lemma.

**Lemma 1.** Let $x_t$ follow a discrete-state Markov process with Poisson jumps between states $\{E_1, E_2, ..., E_N\}$, with transition probabilities $\Pr(x_{t+h} = E_n | x_t = E_k) = \xi_{n,k} h + o(h)$. Then (i) the expected change in $x_t$ is

$$\mathrm{E}(\mathrm{d}x_t | x_t = E_k) = \sum_{i=1}^{N} \xi_{i,k}(E_i - E_k)\,\mathrm{d}t,$$

and (ii) its instantaneous variance is

$$\mathrm{Var}(\mathrm{d}x_t | x_t = E_k) = \sum_{i=1}^{N} \xi_{i,k}(E_i - E_k)^2\,\mathrm{d}t.$$

Proof of Lemma 1: (i) The first jump arrives with arrival rate $\sum_{i=1}^{N} \xi_{i,k}$. Conditional on a jump occurring, the probability of state $n$ is $\xi_{n,k}/\sum \xi$, and the jump size is $(E_n - E_k)$ in this case. Taking expectation over all possible states, we get the result. (ii) Note that the instantaneous volatility equals the instantaneous second moment, $\mathrm{E}[(\mathrm{d}x_t)^2 | x_t = E_k]$, because $\mathrm{E}(\mathrm{d}x_t | x_t = E_k)^2$ is of order $O(\mathrm{d}t^2)$. Then applying (i) to jumps of size $(E_n - E_k)^2$, we obtain the result of the lemma.

Using the lemma, substitute in $E_n = n$ and $\xi_{i,k} = \lambda k$ if $i = k+1$, $\xi_{i,k} = \gamma k$ if $i = k-1$, and $\xi_{i,k} = 0$ otherwise. Then $\mathrm{E}(\mathrm{d}n) = (\lambda - \gamma) n\,\mathrm{d}t$ and $\mathrm{Var}(\mathrm{d}n) = (\lambda + \gamma) n\,\mathrm{d}t$. Divide by $n$ to obtain the result.

## A.3 Proof of Proposition 3

With $\varepsilon = 2$, aggregate output is $Y = NL$. We then have to show that

$$\mathrm{E}(\mathrm{d}N/N) = x\,\mathrm{d}t,$$

independent of $N$ and $t$.

From equation (18),

$$\frac{N(t+h)-N(t)}{N(t)} = \begin{cases} \lambda h & \text{with } \Pr = 1 - O(h) \\ \lambda h - \sum_{i=1}^{\tilde{N}} m_i/N & \text{with } \Pr = \gamma h + o(h) \\ \vdots & \vdots \\ \lambda h - \sum_{i=k}^{\tilde{N}} m_i/N & \text{with } \Pr = \gamma h + o(h) \\ \vdots & \vdots \\ \lambda h - m_{\tilde{N}}/N & \text{with } \Pr = \gamma h + o(h) \end{cases}$$

Taking expectations and using the fact that $\sum_{k=1}^{\tilde{N}} \sum_{i=k}^{\tilde{N}} m_i = N$, we get

$$\mathrm{E}\left[\frac{N(t+h)-N(t)}{N(t)}\right] = (\lambda - \gamma)h + o(h).$$

Taking the limit as $h \to 0$, we obtain the result. The innovation intensity $\lambda$ is independent of $n$ and $t$, and so is the expected growth rate $x$. The solution for $x$ immediately follows from the solution for the firm-level $\lambda$.

## A.4 Proof of Proposition 4

We have to show that

$$\mathrm{Var}(\mathrm{d}N/N) = \gamma \sum_{k=1}^{\infty} s_k^2\,\mathrm{d}t.$$

From equation (18),

$$\left[\frac{N(t+h)-N(t)}{N(t)}\right]^2 =$$

$$\begin{cases} \lambda^2 h^2 & \text{with } \Pr = 1 - O(h) \\ \lambda^2 h^2 - \sum_{i=1}^{\tilde{N}} m_i^2/N^2 - 2\lambda h \sum_{i=1}^{\tilde{N}} m_i/N & \text{with } \Pr = \gamma h + o(h) \\ \vdots & \vdots \\ \lambda^2 h^2 - \sum_{i=k}^{\tilde{N}} m_i^2/N^2 - 2\lambda h \sum_{i=k}^{\tilde{N}} m_i/N & \text{with } \Pr = \gamma h + o(h) \\ & \vdots \\ \lambda^2 h^2 - \sum_{i=k}^{\tilde{N}} m_i^2/N^2 - 2\lambda h m_{\tilde{N}}/N & \text{with } \Pr = \gamma h + o(h) \end{cases}$$

Taking expectations and taking $h \to 0$, we get

$$\mathrm{E}\left(\frac{\mathrm{d}N^2}{N^2}\right) = \gamma \sum_{k=1}^{\tilde{N}} \frac{(\sum_{i=k}^{\tilde{N}} m_i)^2}{N^2}\,\mathrm{d}t = \gamma \sum_{k=1}^{\infty} s_k^2\,\mathrm{d}t,$$

where the last equality follows from the definition of $s_k$. The variance is the same, because

$$\frac{\mathrm{Var}(\mathrm{d}N/N)}{\mathrm{d}t} = \frac{\mathrm{E}(\mathrm{d}N^2/N^2) - \mathrm{E}(\mathrm{d}N/N)^2}{\mathrm{d}t} =$$

$$\gamma \sum_{k=1}^{\infty} s_k^2 - \lim_{h\to 0} \frac{(\lambda-\gamma)^2 h^2}{h} = \gamma \sum_{k=1}^{\infty} s_k^2.$$

## A.5 Proof of Proposition 5

We will show that in an economy where the overall number of varieties is $N$, volatility is bounded from above by $\gamma/N$. Since GDP is linear in $N$, the statement in the proposition follows immediately.

Take an economy with $\tilde{N}$ *distinct* varieties, in which the average firm uses $N \leq \tilde{N}$ varieties. (Recall that the measure of firms is one, so the total number of varieties can be thought of as the *average*.) To simplify notation, define $M_k = \sum_{i=k}^{\tilde{N}} m_i$. Volatility equals $\gamma \sum_{k=1}^{\tilde{N}} s_k^2$, where $s_k = M_k/N$ is the share of variety $k$ in overall GDP. Each variety $k$ is used by at most a unit measure of firms, $M_k \leq 1$. What is the highest possible volatility in this economy conditional on its level of GDP per capita, $N$? Note that this exercise differs from the one discussed on page 25, where we looked at the *unconditional* minimum and maximum of volatility, also changing average GDP at the same time.

We need to find the technology distribution $\{s_k\}$ that maximizes:

$$\max_{\{s_k\}} \gamma \sum_{k=1}^{\tilde{N}} s_k^2$$

$$\text{s.t. } 0 \leq M_k \leq 1$$

$$\sum_{k=1}^{\tilde{N}} M_k = N$$

with $s_k = M_k/N$. The maximum is attained when the first $N$ varieties are used by all firms, $M_k = 1$ for $k = 1, ..., N$ and no other varieties are used by any firms, $M_k = 0$ for all $k = N+1, ..., \tilde{N}$. The maximum volatility is $\gamma \sum_{k=1}^{N} (1/N)^2 = \gamma/N$.

It may seem counterintuitive at first that an even distribution of varieties maximizes volatility. However, this is not an even distribution of *all* $\tilde{N}$ varieties, as those with index higher than $N$ are not used at all. This is in fact the most concentrated distribution of $\tilde{N}$ varieties that is consistent with an average variety use of $N$.

Also note for completeness that a (strict) lower bound on volatility conditional on $N$ is given by $1/N^2$. We solve the minimum problem—$\min_{\{s_k\}} \gamma \sum_{k=1}^{\tilde{N}} s_k^2$ subject to the same constraints as above. Suppose there are $\tilde{N}$ distinct varieties in the economy, the first of which is used by all firms, $M_1 = 1$ by our ranking assumption. Aggregate GDP is then $1 + \sum_{k=2}^{\tilde{N}} M_k = N$. To minimize volatility, all varieties with indexes above 1 are going to be used in the same quantity, $M_2 = (N-1)/\left(\tilde{N}-1\right)$. This leads to a volatility of

$$\frac{1}{N^2} + \frac{(1-1/N)^2}{\tilde{N}-1}.$$

Clearly, this is greater than $1/N^2$ for any finite $\tilde{N}$, hence the lower bound.

# B Technology Adoption under Risk Aversion and Financial Autarky

In this Appendix we discuss technology adoption when agents are risk-averse and risk pooling is not possible. Each firm is owned by a risk-averse individual, whose only source of income is the profit of the firm. Utility exhibits risk aversion with $u' > 0$, $u'' < 0$, $u(0) > -\infty$, $u'(0) < \infty$. These latter assumptions ensure the finiteness of the value of the firm even if there is a positive probability that the firm profits (and hence consumption) eventually become zero.

The value of the firm with $n$ varieties at time $t$ is defined as lifetime expected utility,

$$V(n,t) \equiv \mathrm{E}_t \int_{s=t}^{\infty} e^{-\rho s} u\{A(s)n(s) - g[\lambda(s)]n(s)\} \,\mathrm{d}s. \tag{23}$$

The Bellman equation characterizing the firm's problem is

$$\begin{aligned} \rho V(n,t) = & \max_{\lambda} \{ u[A(t)n - g(\lambda)n] \\ & + \lambda n \left[V(n+1,t) - V(n,t)\right] \\ & + \gamma n \left[V(n-1,t) - V(n,t)\right] \\ & + \lim_{h\to 0} \mathrm{E}\left[V(n,t+h) - V(n,t)\right]/h \}, \end{aligned} \tag{24}$$

$$\text{s.t. } A(t) - g(\lambda) \geq 0. \tag{25}$$

This is the same as (12) with the exceptions that (i) flow utility is a concave function of firm profits, and (ii) we rule out borrowing so that adoption has to be financed from current profits.

**Proposition 6.** Optimal technology adoption intensity, $\lambda(n,t)$ is strictly positive for all $n > 0$ and $t$.

*Proof.* Because $g(0) = 0$, the non-negative profit constraint provides a *positive* upper bound on $\lambda$. If the constraint is binding, $\lambda$ is positive. Otherwise we can use the first-order-condition for optimal adoption,

$$u'[A(t)n - g(\lambda)n]g'(\lambda) = V(n+1,t) - V(n,t). \tag{26}$$

The properties of $u'$ and $g'$ ensure that there will be a unique positive $\lambda$ for each $n$ as long as $V(n+1,t) - V(n,t) > 0$. This condition is easy to verify. It is obvious that $V(n+1,t) \geq V(n,t)$, because the firm can always throw away the additional variety and replicate its profits with $n$ varieties. We can also show that it is strictly better off with more varieties.

The value of a firm with $n$ products is $V(n,t)$ defined by (23). Now calculate a lower bound for the expected discounted utility if the firm adds a variety. Suppose the firm does not change its adoption efforts but keeps them at $\lambda(n)$. Let us denote the value of this strategy by $\tilde{V}()$. It is clear that $V(x,t) \geq \tilde{V}(x,t)$ for all $x$ and $t$, because the firm cannot lose by adjusting its adoption intensity optimally.

Now suppose that the additional variety is useless, $\tilde{V}(n+1,t) = V(n,t)$. In this case the firm does not innovate, and is making profits $A(t)$ per variety. The flow profits the additional variety generates while working are strictly positive, which ensures $\tilde{u}(t) > u(t)$ for all $t \leq T_{n+1}$, because $u' > 0$ even if the consumer is risk averse. Because the new variety is expected to have a positive lifetime ($T_i > 0$ with probability 1), we have that $\tilde{V}(n+1,t) > V(n,t)$, a contradiction. Hence $\tilde{V}(n+1,t) > V(n,t)$ and $V(n+1,t) > V(n,t)$. □

The proof relies on the property that new varieties lead to higher profits. This is why firms have an incentive for technological diversification even in the complete absence of financial markets. Of course, the *magnitudes* may vary with the degree of financial development and technology adoption may be faster or slower in financial developed economies. However, we demonstrated that financial deepening is not *required* for technological diversification to work.

The result that the adoption intensity is positive for *all* $n$ depends on the functional form assumptions about the cost of adoption. In particular, the Inada conditions ensure that it is always optimal to devote some resource to adoption as long as the marginal benefit is positive. Of course, if the marginal cost of adoption is bounded away from zero, there is a range of positive but small marginal benefits for which adoption intensity will be zero. This does not alter the result that financial development is not a *necessary pre-condition* for technological diversification.

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