Aggregation, Liquidity, and Asset Prices with Incomplete Markets

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Abstract

We analytically characterize asset-pricing and consumption behavior in two-account heterogeneous-agent models with aggregate risk. We show that trading frictions can simultaneously explain (1) household-level consumption behavior such as high marginal propensities to consume, (2) a zero-beta rate on equities that satisfies an aggregate consumption Euler equation, (3) a return on safe assets that does not, and (4) a flat securities market line. The return of equities is well explained by aggregate consumption, while the return of safe assets reflects a large and volatile liquidity premium.

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1 Introduction

We propose a theory of asset pricing and consumption behavior based on liquidity frictions. Our starting point is a set of empirical facts in apparent tension with each other:

1. At the household level, consumption behavior is not well described by a simple Euler equation. Households have a high marginal propensity to consume (MPC) from cash transfers, even for households with significant wealth (Johnson et al. (2006); Jappelli and Pistaferri (2010); Kaplan et al. (2014)), but a low MPC from capital gains in the stock market (Chodorow-Reich et al. (2021)).

2. At the aggregate level, consumption satisfies a simple Euler equation with the zero-beta rate, the conditionally expected return on a zero-beta equity portfolio (Di Tella et al. (2023)).

3. Aggregate consumption does not satisfy an Euler equation with safe rates (Hansen and Singleton (1982)). There is a large and volatile spread between zero-beta and safe rates.

4. The securities market line is flat on average: the return of zero-beta portfolios is close to the market return (Black (1972)).

In this paper we show that these facts can be jointly explained by the presence of liquidity frictions: safe assets are liquid and equities are not. "Two-account" models in the style of Kaplan and Violante (2014) propose a precise notion of what liquidity means: there are trading frictions that prevent smooth rebalancing between a liquid and an illiquid account, and all payments must be made out of the liquid account. Kaplan and Violante (2022) show that these liquidity frictions can explain the pattern of household-level asset holdings and marginal propensities to consume (fact 1). We show that the same liquidity frictions can also explain why at the aggregate level the consumption Euler equation holds for *illiquid* assets (fact 2), but not for *liquid* assets (fact 3), provided that the dividend-price ratio is volatile and only weakly predicts near-term dividend growth (in line with Campbell and Shiller (1988) and Cochrane (2008)). An aggregate Euler equation holds for illiquid assets even though each individual household's consumption behavior is the result of a complex interaction between borrowing constraints, idiosyncratic risks, and aggregate risks. We also show that in spite of these complexities, a simple Consumption CAPM holds in our model, consistent with small risk premia and a flat securities market line (fact 4).

Our results suggest that the return of equities is well explained by aggregate consumption, with a large and time-varying zero-beta rate and a small risk premium, while the return of safe assets mostly reflects a large and volatile liquidity premium.

The Model. We work with a canonical two account heterogeneous-agent model designed to explain household-level consumption and asset-holding behavior,[1] and we extend it to incorporate aggregate shocks. Only a fraction of capital income can be used to back liquid assets that can be used to pay for consumption. The rest must be held in an illiquid account that can only be rebalanced when a trading opportunity arises with Poisson intensity and/or at a fixed cost. The spread between liquid and illiquid assets compensates households for the risk of running out of liquid assets and having to live hand-to-mouth.

In our model the key difference between liquid and illiquid assets is that the former can be used to pay for consumption. For this reason, we view equities as illiquid, and money-like instruments as liquid (consistent with high MPCs from cash transfers and low MPCs from equity wealth). Treasury bonds, like equities, cannot be directly used to make payments, but can be used by financial intermediaries to back payment instruments such as bank deposits and money markets. For this reason, we view them as liquid.[2] However, we don't explicitly model the supply of liquid assets by financial intermediaries and we don't distinguish between assets with varying degrees of liquidity (e.g. checking and savings accounts and money markets).

A key technical contribution of the paper is to solve this heterogeneous-agent model with aggregate shocks in closed form. This is important when studying asset-pricing features that would be eliminated by linearization. Our results build on the work of Krueger and Lustig (2010) and Werning (2015), but we account for many features essential to the question we tackle: trading frictions, assets in positive supply (both liquid and illiquid), intertemporal elasticity below one, and general stochastic processes for idiosyncratic labor income and aggregate output. To obtain exact aggregation we incorporate a form of counter-cyclical labor income risk (e.g. it is harder to find a job during recessions). This combination of assumptions allows us to analytically characterize asset prices in a heterogeneous-agent economy with aggregate risk, without imposing any restrictions on the details of the income process and frictions that govern the counterfactual steady-state economy.[3] Our solution method involves introducing a stochastic time change that allows us to represent the economy with aggregate shocks in terms of the economy without aggregate shocks.

Our exact aggregation results, while important to properly characterize asset prices, hinge on the absence of redistributive effects for aggregate shocks. In our view, the distributive

[1] e.g. Kaplan and Violante (2014), Kaplan et al. (2018), and Auclert et al. (2023)

[2] Empirically, the return on payment instruments is almost exactly proportional to that of Treasurys so fact 3 applies equally to both.

[3] That is, our results apply to the steady-state economies studied by e.g. Kaplan and Violante (2014), Kaplan et al. (2018), and Auclert et al. (2023). They also cover, as a special case, steady-state heterogenous agent models with a single account (as in Aiyagari (1994) and the literature that followed).

effects of aggregate shocks are likely to play an important role in the explanation of some economic phenomena. We view our analytical results as a theoretical benchmark to understand more complex models that incorporate more meaningful distributional effects.

Main Results. We start with a setting with log preferences and show that while there is a spread between the return of the liquid and illiquid assets, this spread is invariant to aggregate shocks. The return of both assets satisfies an aggregate consumption Euler equation (with different intercepts) and dividend-price ratios are constant.

We then consider the CRRA case with intertemporal elasticity below one. We derive relations between expected aggregate consumption growth and returns on both liquid and illiquid assets. The illiquid asset is not very good at insuring idiosyncratic labor income risk because households may not be able to access it when they really need it, so the aggregate consumption Euler equation holds as a good approximation. Liquid assets, on the other hand, allow agents to self insure against idiosyncratic labor income risk, so they carry a liquidity premium. The liquidity premium moves with the dividend-price ratio. When asset prices are high relative to output, the supply of liquidity is high relative to the demand for it and the liquidity premium is small. With low intertemporal elasticity this happens when expected consumption growth going forward is low, producing a pro-cyclical liquidity premium. In addition, even though agents face uninsurable risk and trading frictions, a simple Consumption CAPM explains risk premia in the model.

We derive a sufficient-statistic expression for asset returns that allows us to quantitatively evaluate the mechanism in the model while abstracting from many microeconomic details. Our model can quantitatively match the behavior of zero-beta rates, safe rates and market returns, as long as the dividend-price ratio is volatile and has limited ability to predict growth. Under these conditions, our model provides a new perspective on classic asset-pricing puzzles: the equity premium, the flat securities market line, the relatively stable risk-free rate, and equity volatility, and is consistent with evidence on return predictability.

Related Literature. Our work is most closely related to Krueger and Lustig (2010) and Werning (2015). Krueger and Lustig (2010) study asset prices in a "one-account" heterogeneous agent model in which there are shocks to the level of aggregate output, but not to its growth rate (aggregate output and consumption growth are I.I.D.). They consider the case of CRRA preferences and show that the standard "consumption CAPM" holds despite the presence of uninsurable idiosyncratic income risk and borrowing constraints. The key proof technique involves constructing a steady-state equilibrium without aggregate risk, and then constructing the equilibrium with aggregate risk using that steady-state equilibrium.

The Krueger and Lustig (2010) model has little to say about the consumption Euler equation directly, as interest rates are constant as a consequence of the I.I.D. growth shocks. We recover their "consumption CAPM" result in a more general setting with time-varying expected growth and trading frictions, which allows us to study the aggregate consumption Euler equation for the zero-beta rate and safe bond rates.

Our paper and Werning (2015) adapt the technique of Krueger and Lustig (2010) to consider the case in which consumption growth is not constant over time. Werning (2015) studies three different versions of the log utility case in which the standard aggregate Euler equation holds: (i) a case with zero asset supply, (ii) a case in which initial bond holdings are zero, and (iii) an RBC model with fully-depreciating capital (as in the Brock and Mirman (1972) model).[4] The first two of these cases involve models with no aggregate risk (the aggregate shocks is a one-time, unanticipated shock), and all of these cases are one-account models. Our results hold in a two-account model in which assets are positive supply and there is aggregate risk. In this sense, they extend the results of Werning (2015). They are closest, conceptually, to case (ii) above, in that the aggregate shocks in our model do not redistribute wealth across agents.

The extension of the Werning (2015) results on the log utility case to the two-account, positive-asset supply case with aggregate risk brings those results closer to the quantitative HANK models. However, log utility is a non-starter for the purposes of asset pricing, primarily because it leads to constant price-dividend ratios. In fact, it is exactly this property that ensures that the assumption of acyclical labor income risk that Werning (2015) emphasizes leads to acyclical human capital values, which is the key to aggregation in the log case. A second contribution of our paper is to provide conditions for the CRRA case involving cyclical labor income risk that lead to acyclical human capital values, and hence aggregation. This step is what allows us to provide analytical results for the CRRA case in our model.

More importantly, our results for the CRRA case allow us to study a case in which the spread between liquid and illiquid assets is not constant, and in fact quite volatile. The third key contribution of our paper is to show that two-account heterogeneous-agent models with CRRA preferences can explain why the aggregate Euler equation holds for illiquid assets but not liquid assets, consistent with the evidence of Di Tella et al. (2023). The model can also generate high (on average) equity returns with moderate levels of risk-aversion, and at the same time explain why illiquid assets that are more exposed to aggregate risk (high "beta" assets) do not offer higher returns than illiquid assets that are less exposed (lower "beta" assets). That is, the model generates a flat "security market line," consistent with empirical evidence (Black et al. (1972); Frazzini and Pedersen (2014); Hong and Sraer (2016)). The

[4]Werning (2015) has some results for the zero asset supply case that apply in the CRRA as well.

key to these results is the large spread between liquid and illiquid assets generated by the two-account model. That is, the two-account model leads naturally to a theory of asset pricing that emphasizes liquidity instead of risk, in the spirit of Bansal and Coleman (1996) and Lagos (2010). We treat bonds as liquid, perhaps because they can be used by banks to back deposits (Diamond, 2020), or because they are held in zero-maturity money market funds. But we don't model the liquidity of safe bonds in detail, or the difference between money markets, saving accounts, and checking accounts.

Our work, which delivers analytical asset pricing results in heterogeneous-agent models, is part of a literature that attempts to study heterogeneous-agent models with aggregate shocks analytically. Acharya and Dogra (2020) and Acharya et al. (2023) provide results for the CARA-normal setting without borrowing constraints. Bilbiie (2008) derives analytical results for economies in which agents have two types ("TANK" models). Ravn and Sterk (2021) and Challe (2020) provide results that (like many of the results in Werning (2015)) apply to an economy with zero-liquidity. We innovate, relative to this literature, by providing results for the canonical two-account setting, with standard preferences, general forms of heterogeneity, and positive liquidity. That said, many of the aforementioned papers incorporate standard New Keynesian features (e.g. price stickiness) that have not yet been incorporated into our framework.

Our work complements recent efforts to develop numerical techniques capable to studying asset pricing in these models (Bhandari et al., 2023; Bilal, 2023). Relative to these efforts, our approach has the advantage of sharp and intuitive analytical results, and the disadvantage of requiring particular assumptions that limit the redistributive effects of aggregate shocks in the model. We view the two approaches as complementary: our results show the assumptions required to generate redistributive effects, and the approaches of Bhandari et al. (2023) and Bilal (2023) show how quantitative results can be obtained in the presence of these effects.

The structure of the paper is as follows. We begin in Section 2 with the log utility case. In Section 3 we extend our main results to the CRRA case. In Section 4 we characterize risk premia. In Section 5 we show via a back-of-the-envelope quantification that the CRRA case can generate an aggregate Euler equation that holds approximately for the illiquid but not the liquid assets. In Section 6 we extend our results to the case of transition dynamics. In Section 7 we conclude.

2 The Log Economy

We first introduce the benchmark model with log preferences. Our analysis in this section follows closely the arguments of Krueger and Lustig (2010) and in particular Werning (2015),

applied to the two-account models in the style of Kaplan and Violante (2014).

2.1 Preferences and Technology

There is a continuum of households $i \in [0, 1]$ with identical preferences:

$$U(C_i) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log(C_{it}) dt\right]. \tag{1}$$

Aggregate output evolves according to the following stochastic process:

$$\frac{dY_t}{Y_t} = g_t dt + \sigma_Y(Y_t, g_t) \cdot dM_t, \tag{2}$$

$$dg_t = \mu_g(g_t, Y_t)dt + \sigma_g(g_t, Y_t) \cdot dM_t, \tag{3}$$

where M_t is a vector of independent standard Brownian motions. In Sections 2 and 3 we focus on the special case where $\sigma_Y(Y_t, g_t) = 0$, so that there are shocks to the growth rate but not the the level of output. This eliminates risk premia and allows us to focus on the relationship between interest rates and consumption growth. We return to the general case and characterize risk premia in Section 4. We assume throughout that Y_t is bounded below away from zero.

To simplify our exposition, we study an economy in which output (Y_t) is exogenous and equal to aggregate consumption (C_t). We do not take a stand on what drives movements in output. Output could change due to exogenous changes in TFP, but could also result from changes in employment, capital utilization, etc. We assume that the capital stock is fixed (i.e. there is no investment), and that factor shares are constant. Specifically, a fraction α of output accrues to capital owners and a fraction $(1 - \alpha)$ to workers.[5]

Households face uninsurable risk to their labor income. Household i's idiosyncratic state is a vector e_{it} whose evolution is governed by an infinitesimal generator $\mathcal{L}_e$. Its share of aggregate labor income, denoted e^0_{it}, is a function of e_{it}.[6] We assume throughout that e^0_{it} is bounded below away from zero. For now, we also assume $\mathcal{L}_e$ is invariant to aggregate shocks.

[5]This can be motivated by a Cobb-Douglas production function in conjunction with the assumption that factors earn their marginal products.

[6]In the simplest case, e_{it} is a scalar and $e^0_{it} = e_{it}$ but the general notation can accommodate cases where, for instance, idiosyncratic productivity has more- and less-persistent components, or separate productivity and employment status.

Aggregate resource constraints are:[7]

$$\int_0^1 C_{it} di = C_t = Y_t, \tag{4}$$

$$\int_0^1 e_{it}^0 di = 1. \tag{5}$$

2.2 Market Incompleteness and Trading Frictions

Households have two accounts: an illiquid account, with a balance denoted by A_{it}, and a liquid account, with a balance denoted by B_{it}. Their dynamic budget constraints are:

$$dA_{it} = r_{at} A_{it} dt + D_{it} dN_{it} + A_{it} \sigma_{at} \cdot dM_t, \tag{6}$$

$$dB_{it} = \left(r_{bt} B_{it} + e_{it}^0 (1-\alpha) Y_t - C_{it} + H_t \right) dt - \left(D_{it} + \mathbb{I}_{D_{it} \neq 0} \kappa B_t \right) dN_{it} + B_{it} \sigma_{bt} \cdot dM_t, \tag{7}$$

together with the borrowing constraints $A_{it}, B_{it} \geq 0$. The accounts have expected returns, r_{at} and r_{bt} respectively, and may also be exposed to aggregate risk through σ_{at} and σ_{bt}. Households pay for consumption out of the liquid account, and receive their labor income $e_{it}^0 (1-\alpha) Y_t$ into it. Households can only move funds between the illiquid and liquid accounts at idiosyncratic trading opportunities that arrive at a Poisson rate χ. When an opportunity arrives, they must also pay a fixed fee κ if they choose to rebalance. The fee is denominated in units of the liquid asset, so it is scaled by the total value of the liquid asset, denoted B_t. D_{it} are the funds moved into the illiquid account conditional on the arrival of a trading opportunity and N_{it} is the Poisson counting process associated with household i's trading opportunities. For simplicity we assume the rebalancing fees $H_t = \chi \left(\int_0^1 \mathbb{I}_{D_{it} \neq 0} di \right) \kappa B_t$ are uniformly rebated lump-sum to households.[8]

Household i's problem is to pick processes (C_{it}, D_{it}) to maximize utility (1) subject to budget constraints (6) and (7) and the no-borrowing constraints, taking processes $(Y, g, r_a, r_b, \sigma_a, \sigma_b)$ as given.

2.3 Supply of Liquid and Illiquid Assets

Both liquid and illiquid assets are ultimately claims on capital. A fraction θ of capital income accrues to the owners of liquid assets and the remaining $1-\theta$ to owners of illiquid assets.

[7] Equation (5) involves only exogenous objects, so it is really a consistency condition on $\mathcal{L}_e$.

[8] These are the trading frictions in Kaplan and Violante (2022), except we denominate them in units of the liquid asset so they don't appear in the resource constraint. They nest a pure Poisson trading friction and a pure fixed cost as special cases. In the Appendix we explore alternative formulations.

The value of each asset is then:[9]

$$A_t = \mathbb{E}_t \left[\int_t^\infty e^{-\int_t^s r_{au} du} (1-\theta) \alpha Y_s ds \right], \tag{8}$$

$$B_t = \mathbb{E}_t \left[\int_t^\infty e^{-\int_t^s r_{bu} du} \theta \alpha Y_s ds \right]. \tag{9}$$

Although the cash flows are proportional to each other, the discount rates r_{at} and r_{bt} are different and can move differently in response to aggregate shocks. Consistent with the budget constraints (6) and (7), the expected return from holding each asset is r_{at} and r_{bt}, and the volatility of A_t and B_t pin down σ_{at} and σ_{bt}.[10] The parameter θ represents the fraction of capital income that can ultimately be used to back liquid assets.

The price-dividend ratios for each asset are:

$$P_{at} = \frac{A_t}{(1-\theta)\alpha Y_t}, \quad P_{bt} = \frac{B_t}{\theta \alpha Y_t}. \tag{10}$$

Market clearing for assets requires:

$$\int_0^1 A_{it} di = A_t, \quad \int_0^1 B_{it} di = B_t. \tag{11}$$

2.4 State Space and Generator

It is useful to normalize variables as follows:

$$c_{it} = C_{it}/Y_t, \quad a_{it} = A_{it}/A_t, \quad b_{it} = B_{it}/B_t, \quad d_{it} = D_{it}/B_t, \quad h_t = H_t/Y_t$$

An advantage of this normalization is that a_{it} and b_{it} are the shares of the assets that each household owns, and are therefore not affected by aggregate shocks on impact (in contrast to the value of those asset positions, A_{it} and B_{it}). That is, there are no diffusion terms in the evolution of a_{it} and b_{it}.

Lemma 1. *In the absence of rebalancing ($dN_{it} = 0$):*

$$da_{it} = \frac{a_{it}}{P_{at}} dt, \quad db_{it} = \frac{1}{\alpha \theta P_{bt}} \left(\alpha \theta b_{it} + e^0_{it}(1-\alpha) - c_{it} + h_t \right) dt.$$

[9] Equations(8) and (9) assume that there are no bubbles. The model can be straightforwardly extended to the case where r_b is below the growth rate of the economy and no dividends accrue to the liquid asset, which is relevant for our quantitative exercises.

[10] For now there is only one asset in each account, so the volatilities σ_{at} and σ_{bt} in the budget constraints (6) and (7) are the ones derived from (8) and (9). In Section 4 we will allow households to trade a complete set of securities in each account to characterize risk premia.

See the Appendix for all proofs. Lemma 1 describes how the household's shares in total illiquid and liquid assets evolve when the household does not rebalance between accounts. Dividends from each asset are paid into the corresponding account. For the illiquid account, no rebalancing means that dividends are reinvested, and as a result share ownership for these households grows at the rate of the dividend yield (P_{at}^{-1}). Rebalancing households must therefore (by market clearing) be net sellers of the illiquid asset; the price of the illiquid asset will adjust to ensure this is the case. For the liquid account, the same logic applies except that there are additional terms that capture labor income being paid in and consumption being paid out. In both cases the evolution is deterministic: aggregate shocks may affect asset values but all holders of each asset share this risk pro-rata, so share holdings (unlike the value of the assets held) are unaffected by aggregate shocks.

Households' optimal policies depend on the history of aggregate shocks and on their idiosyncratic state, $c_{it} = c_t(a_{it}, b_{it}, e_{it})$ and $d_{it} = d_t(a_{it}, b_{it}, e_{it})$. Given these policy functions, the evolution of (a_{it}, b_{it}, e_{it}) can be described by the infinitesimal generator $\mathcal{L}_{abe}(c_t, d_t; h_t, P_{at}, P_{bt})$, defined for arbitrary test function $f(a, b, e)$:

$$\mathcal{L}_{abe}(c_t, d_t; h_t, P_{at}, P_{bt})f(\cdot) = \mathcal{L}_e f(\cdot) + \frac{a}{P_{at}} f_a(\cdot) + \frac{1}{\alpha\theta P_{bt}}(\alpha\theta b + e^0(1-\alpha) - c_t(\cdot) + h_t) f_b(\cdot) \quad (12)$$

$$+\chi\bigg(f\Big(a + \frac{\theta P_{bt}}{(1-\theta)P_{at}} d_t(\cdot), b - d_t(\cdot) - \kappa \mathbb{I}_{d_t(\cdot)\neq 0}, e\Big) - f(\cdot)\bigg).$$

The first term captures the evolution of e_{it}, the second and third the evolution of a_{it} and b_{it} in the absence of a trading opportunity, and the last term captures rebalancing across accounts. The generator $\mathcal{L}_{abe}(c_t, d_t; h_t, P_{at}, P_{bt})$ depends on the history of aggregate shocks through h_t, P_{at}, P_{bt}, and the policy functions $c_t(\cdot)$ and $d_t(\cdot)$.

Let $\mu_t(a, b, e)$ be the population measure of households' idiosyncratic state. Its evolution can be described with a KFE:

$$d\mu_t(\cdot) = \mathcal{L}^{\dagger}_{abe}(c_t, d_t; h_t, P_{at}, P_{bt})\mu_t(\cdot)dt, \quad (13)$$

where $\mathcal{L}^{\dagger}_{abe}(c_t, d_t; h_t, P_{at}, P_{bt})$ is the adjoint operator of $\mathcal{L}_{abe}(c_t, d_t; h_t, P_{at}, P_{bt})$.[11] Aggregate shocks can affect the evolution of μ_t through the operator $\mathcal{L}^{\dagger}_{abe}(c_t, d_t; h_t, P_{at}, P_{bt})$, but they don't by themselves change the distribution μ_t on impact, because of the way we have normalized the state variables.

[11] The adjoint operator and the "KFE" equation should be understood in the weak*-sense; in two-account models, μ_t will often not be absolutely continuous with respect to the Lebesgue measure.

The total rebalancing fees, normalized by aggregate output, are

$$h_t = \chi \left(\int \mathbb{I}_{d_t(\cdot)\neq 0} d\mu_t(\cdot) \right) \kappa P_{bt}. \tag{14}$$

We will also use the generator for Y and g (here we use that $\sigma_Y(Y_t, g_t) = 0$):

$$\mathcal{L}_{Yg} f(\cdot) = f_Y(\cdot) gY + f_g(\cdot)\mu_g(Y,g) + \frac{1}{2} f_{gg}(\cdot)\sigma_g(Y,g)^2.$$

Note that here and throughout the paper, we use the square of a vector to indicate the ℓ^2-norm e.g. $\sigma_g(Y,g)^2 = \sigma_g(Y,g)^T \cdot \sigma_g(Y,g)$.

2.5 Competitive Equilibrium

For an initial measure μ_0, a *competitive equilibrium* is a set of adapted policy functions $(c_t^*(\cdot), d_t^*(\cdot))$, price and return processes $(r_{at}^*, \sigma_{at}^*, P_t^{a*}, r_t^{b*}, \sigma_{bt}^*, P_t^{b*})$, a measure μ_t^* and fees h_t such that (1) policies are optimal in households' problem, (2) μ_t^* satisfies KFE (13) with initial condition $\mu_0^* = \mu_0$, (3) prices and returns satisfy (8) and (9) and h_t satisfies (14), and (4) markets clear:

$$\begin{aligned} \int_{\text{supp}(\mu_t^*)} b d\mu_t^*(a,b,e) &= 1, \\ \int_{\text{supp}(\mu_t^*)} a d\mu_t^*(a,b,e) &= 1, \\ \int_{\text{supp}(\mu_t^*)} c_t^*(a,b,e) d\mu_t^*(a,b,e) &= 1. \end{aligned} \tag{15}$$

In the absence of aggregate shocks, $Y_t = 1$, $g_t = \sigma_Y = \mu_g = \sigma_g = 0$, we can define a *steady state competitive equilibrium* as a set of policy function $(\bar{c}(\cdot), \bar{d}(\cdot))$, prices ($\bar{r}_a$, $\bar{r}_b$, $\bar{P}_a$, $\bar{P}_b$), a measure $\bar{\mu}$ and fees $\bar{h}$ such that (1) policies are optimal in households' problem, (2) $\bar{\mu}$ satisfies the KFE (13) with initial condition $\mu_0 = \bar{\mu}$, (3) $\bar{P}_a = \bar{r}_a^{-1}$, $\bar{P}_b = \bar{r}_b^{-1}$, and $\bar{h} = \chi \int \mathbb{I}_{\bar{d}(\cdot)\neq 0} d\bar{\mu}(\cdot)\kappa \bar{P}_b$, and (4) markets clear analogously to (15).

2.6 Steady State Equilibrium

We start with the steady state equilibrium. The value function for a household is $\bar{V}(a,b,e)$ and the HJB equation is:

$$\rho \bar{V}(a,b,e) = \max_{\substack{c\geq 0, \mathbb{I}_{b=0} c \leq (1-\alpha)e^0 + \bar{h}, \\ d \in [-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b, b]}} \ln c + \mathcal{L}_{abe}(c,d;\bar{h},\bar{P}_a,\bar{P}_b)\bar{V}(a,b,e). \tag{16}$$

If $b = 0$, then consumption is limited by current income, $c \leq (1-\alpha)e^0 + \bar{h}$. The policy functions $\bar{c}$ and $\bar{d}$ are derived from the the HJB, and the KFE is:

$$\bar{\mathcal{L}}^{\dagger}_{abe}(\bar{c}, \bar{d}; \bar{h}, \bar{P}_a, \bar{P}_b)\bar{\mu} = 0. \tag{17}$$

Finding a steady state equilibrium requires finding $\bar{h}$, $\bar{P}_a$, and $\bar{P}_b$ such that the policy functions $\bar{c}$ and $\bar{d}$ derived from (16) and the measure $\bar{\mu}$ derived from (17) satisfy market clearing and $\bar{h} = \chi \int \mathbb{I}_{\bar{d}(\cdot)\neq 0} d\bar{\mu}(\cdot)\kappa\bar{P}_b$. In what follows, we will simply assume that a steady state equilibrium can be found.

2.7 Competitive Equilibrium with Aggregate Shocks

We conjecture a competitive equilibrium with aggregate shocks that is Markov in (Y, g) and where $P^*_{at} = \bar{P}_a$, $P^*_{bt} = \bar{P}_b$, and $h_t = \bar{h}$. Given this conjecture, the generator is the same as in the steady state, conditional on the policy functions:

$$\mathcal{L}_{abe}(c, d; h^*_t, P^*_{at}, P^*_{bt}) = \mathcal{L}_{abe}(c, d; \bar{h}, \bar{P}_a, \bar{P}_b).$$

The value function is $V(a, b, e; Y, g)$, and the HJB is:

$$\rho V(a,b,e;Y,g) = \max_{\substack{c\geq 0, \mathbb{I}_{b=0} c \leq (1-\alpha)e^0 + \bar{h}, \\ d \in [-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b, b]}} \ln(Y) + \ln c + \mathcal{L}_{abe}(c, d; \bar{h}, \bar{P}_a, \bar{P}_b)V(a,b,e;Y,g) + \mathcal{L}_{Yg}(Y,g)V(a,b,e;Y,g). \tag{18}$$

We can then guess and verify that:

$$V(a, b, e; Y, g) = \bar{V}(a, b, e) + \phi(Y, g).$$

Under the guess, the HJB simplifies to:

$$\rho\bar{V}(a,b,e) + \rho\phi(Y,g) = \max_{\substack{c\geq 0, \mathbb{I}_{b=0} c \leq (1-\alpha)e^0 + \bar{h} \\ d \in [-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b, b]}} \ln(Y) + \ln c + \mathcal{L}_{abe}(c, d; \bar{h}, \bar{P}_a, \bar{P}_b)\bar{V}(a,b,e) + \mathcal{L}_{Yg}(Y,g)\phi(Y,g).$$

Using the steady state HJB (16) to cancel out terms, we get:

$$\rho\phi(Y, g) = \ln(Y) + \mathcal{L}_{Yg}(Y, g)\phi(Y, g), \tag{19}$$

with solution:

$$\phi(Y, g) = \mathbb{E}\left[\int_t^{\infty} e^{-\rho(s-t)} \ln(Y_s) ds | Y_t = Y, g_t = g\right], \tag{20}$$

which verifies our guess. The policy functions c and d are therefore the same as in the steady state equilibrium. This means that, if we start at $\mu_0 = \bar{\mu}$,

$$\mathcal{L}^{\dagger}_{abe}(c^*_t, d^*_t; h^*_t, P^*_{at}, P^*_{bt})\mu_t = \mathcal{L}^{\dagger}_{abe}(\bar{c}, \bar{d}; \bar{h}, \bar{P}_a, \bar{P}_b)\bar{\mu} = 0,$$

so the measure remains at $\mu_t = \bar{\mu}$. Since both policy functions and the measure μ_t are as in the steady state equilibrium, we have market clearing and from (14) we confirm that $h^*_t = \bar{h}$. It only remains to pin down the returns r_{at}, r_{bt}, σ_{at}, and σ_{bt}. For the illiquid asset,

$$\underbrace{\mathbb{E}_t\left[\int_t^{\infty} e^{-\int_t^s r_{au}du}(1-\theta)\alpha Y_s ds\right]}_{A_t} = \bar{P}_a(1-\theta)\alpha Y_t.$$

Taking time derivatives on both sides we get:

$$\left(r_{at}A_t - (1-\theta)\alpha Y_t\right)dt + A_t\sigma_{at}\cdot dM_t = \bar{P}_a(1-\theta)\alpha Y_t g_t dt.$$

We immediately get $\sigma_{at} = 0$. Dividing throughout by $A_t = \bar{P}_a(1-\theta)\alpha Y_t$ and using $\bar{P}_a = \bar{r}_a^{-1}$, we obtain $r_{at} = \bar{r}_a + g_t$. The same argument works for the liquid asset, so $r_{bt} = \bar{r}_b + g_t$.

Proposition 2. *Assume $\sigma_Y(Y_t, g_t) = 0$ and that there exists a steady state equilibrium $(\bar{c}, \bar{d}, \bar{r}_a, \bar{r}_b, \bar{P}_a, \bar{P}_b, \bar{\mu}, \bar{h})$ with C^1 value function $\bar{V}(a, b, e)$ satisfying the HJB equation (16). Then if $\mu_0 = \bar{\mu}$ and and $\phi(Y, g)$ as defined in (20) is finite, there exists a competitive equilibrium with*

$$c^*_t(\cdot) = \bar{c}(\cdot), \quad d^*_t(\cdot) = \bar{d}(\cdot), \quad P^*_{at} = \bar{P}_a, \quad P_{bt} = \bar{P}_b, \quad \mu^*_t = \bar{\mu}, \quad h^*_t = \bar{h}.$$

There is an aggregate consumption Euler equation for both assets $j \in \{a, b\}$,

$$r_{jt} = \bar{r}_j + g_t, \tag{21}$$

and assets are locally safe, $\sigma_{at} = \sigma_{bt} = 0$.

This result allow us to construct the competitive equilibrium with aggregate shocks assuming the initial distribution is $\mu_0 = \bar{\mu}$. In Section 6 we extend this result to construct the competitive equilibrium with aggregate shocks for any initial distribution μ_0.

Discussion of the Log Economy

Exact aggregation is possible because the aggregate shocks do not redistribute wealth. There is no redistribution between capital and labor (fixed capital share α) or between liquid and illiquid assets (fixed θ). In addition, there is no redistribution between labor-endowment types e_{it}. This last property is due to the combination of log preferences and the invariance of the generator $\mathcal{L}_e$ to the aggregate state. Consider two households, one who is currently employed and one who is not. The unemployed one has a backloaded labor income profile relative to the employed one. When a shock lowers g_t, all households expect lower labor income in the future, but thanks to log preferences the interest rates fall one-for-one with g_t, so there is no redistribution (in a net-present-value sense) between back-loaded and front-loaded labor income profiles. In addition, the distribution of that labor income between the two households is unchanged because the generator that governs labor market dynamics, $\mathcal{L}_e$, is independent of the aggregate state of the economy. This last condition—that the process governing idiosyncratic income shares is unaffected by aggregate risk—is the same condition used by Werning (2015) to derive the standard aggregate Euler equation in his analysis of the log utility case.

The log economy generates a spread between liquid and illiquid assets and an aggregate consumption Euler equation for illiquid assets. This spread reflects the convenience of liquid assets, which are necessary to sustain consumption after periods of low labor income. To see this, abstract from the fixed cost ($\kappa = 0$) and consider a household with a trading opportunity who can decide to increase their consumption and reduce their holding of liquid or illiquid assets. For the illiquid asset, the cost is the return until the next trading opportunity, τ_a. For the liquid asset, the return until either the next trading opportunity τ_a or the point at which the household runs out of liquid assets, τ_b . They could reduce their consumption before that, but they cannot guarantee postponing the reduction in consumption beyond that point. We can then write the following household-level Euler equations:

$$C_{it}^{-\gamma} = \mathbb{E}_t\left[e^{\int_t^{\tau_a}(r_{as}-\frac{1}{2}\sigma_{as}^2)dt+\int_t^{\tau_a}\sigma_{as}dM_s}C_{i\tau_a}^{-\gamma}\right] = \mathbb{E}_t\left[\int_t^{\infty}\chi e^{-\chi(u-t)}\times e^{\int_t^{u}(r_{as}-\frac{1}{2}\sigma_{as}^2)dt+\int_t^{u}\sigma_{as}dM_s}C_{iu}^{-\gamma}du\right], \tag{22}$$

$$C_{it}^{-\gamma} = \mathbb{E}_t\left[e^{\int_t^{\tau_a\wedge\tau_b}(r_{bs}-\frac{1}{2}\sigma_{bs}^2)dt+\int_t^{\tau_a\wedge\tau_b}\sigma_{bs}dM_s}C_{i\tau_a\wedge\tau_b}^{-\gamma}\right]. \tag{23}$$

Relative to the illiquid asset, the liquid one provides insurance against the event $\tau_b < \tau_a$ that the household runs out of liquidity before it can access its illiquid assets, which is a high marginal-utility state. If this is guaranteed not to happen, $P(\tau_b < \tau_a) = 0$, then the household will never hold liquid assets (at the margin) as long as there is a positive spread. Households reduce their liquid holdings up to the point that a positive probability of running

out of liquid funds compensates for the higher return of illiquid assets. In equilibrium the spread is constant because the supply of liquid assets is in constant relation to the liquidity needs derived from idiosyncratic labor income shocks and trading opportunities.

The last expression in (22) also shows that the return of the illiquid asset can be interpreted as an intertemporal price. The illiquid asset can be thought of as a security whose payoffs follow an exponential decay. Its return is therefore an exponentially-weighted average of intertemporal prices between time t and $t+u$ (analogous to how the return of an asset with a one-year maturity is the intertemporal price between t and $t+1$). Expression (23) shows this interpretation is not valid for the liquid asset. Its payoff structure depends on the realization of idiosyncratic labor income risk *of the holder* through τ_b. In this sense we can think of it as providing insurance.

The log economy has important shortcomings:

1. The spread between liquid and illiquid assets is constant, and as a result an aggregate Euler equation holds for both the liquid and illiquid assets, contradicting our Fact 3.

2. Estimates of the intertemporal elasticity using the aggregate Euler equation fall well below one. An intertemporal elasticity of 1 is rejected, so the log economy can fit Fact 2 qualitatively but not quantitatively.

3. Price-dividend ratios are constant in the model, whereas in reality they are quite volatile. The log economy is thus a non-starter for the purpose of explaining asset prices.

These failures motivate us to extend the model to CRRA preferences with intertemporal elasticity below 1.

3 The CRRA Economy

We modify the economy of the previous section in two ways. First, households have CRRA preferences:

$$U(C_i) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{C_{it}^{1-\gamma}}{1-\gamma}dt\right].$$

We are interested in the $\gamma > 1$ case. Second, the generator for idiosyncratic labor endowment and the trading friction now depend on the aggregate state of the economy. The generator is now:[12]

$$x_t^{-1}\mathcal{L}_e, \tag{24}$$

[12]This is still consistent with the aggregate resource constraint (5).

and the arrival rate of a trading opportunity is:

$$x_t^{-1}\chi, \tag{25}$$

where:

$$x_t = x(Y_t, g_t) = \rho \frac{1}{Y_t^{1-\gamma}} \mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)} Y_s^{1-\gamma} ds | Y_t, g_t\right]. \tag{26}$$

The object $x(Y, g)$ happens to be the price-dividend ratio in a representative-agent economy, normalized by ρ.[13] Our model does not distinguish between a claim to aggregate consumption and a claim to dividends, so we will interpret x_t as both a wealth-to-consumption ratio and a price-dividend ratio. With $\gamma = 1$ (log), $x(Y, g) = 1$ and we have the setting in the previous section. With $\gamma > 1$, $x(Y, g)$ is high when growth is expected to be low (e.g. during the contraction phase of the cycle). The reason is that with $\gamma > 1$ interest rates fall more than one-for-one with expected growth g_t. We will assume that the stochastic process for (Y_t, g_t) is such that $x(Y_t, g_t)$ is finite for all g_t and $Y_t > 0$.

Adjusting the generator and trading frictions by this precise amount is the condition needed to obtain exact aggregation, by eliminating any distributional effect of aggregate shocks on impact. Of course, redistribution is likely to be central to the explanation of some economic phenomena. Our model is in effect setting up a baseline where redistribution is absent. Our assumptions also have an economic meaning. Expressions (24) and (25) say that during contractions the process for idiosyncratic risk slows down. This is disproportionately bad for those who currently have low labor income or who hold mainly illiquid assets, who will tend to stay in their current state for longer. For example, currently unemployed households are disproportionately less likely to find a job during a recession.[14] Likewise, during contractions liquidity frictions become more severe. While we view both of these assumptions as directionally reasonable, we are imposing a precise functional form, without any free parameters, for the purpose of obtaining exact aggregation.

[13] To see this, write $x_t = \rho\mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)} \left(\frac{Y_s}{Y_t}\right)^{-\gamma} \frac{Y_s}{Y_t} ds | Y_t, g_t\right]$, and notice that in the representative-agent case the SDF is $e^{-\rho t} Y_t^{-\gamma}$.

[14] The generator pertains to each household's *share* of aggregate labor income, e_{it} not the aggregate level of labor income $(1-\alpha)Y_t$. If every households' income is expected to be lower in the same proportion, the generator $\mathcal{L}_e$ would not be affected. The model also doesn't distinguish lower aggregate labor income from lower wages or from lower aggregate employment/higher unemployment.

3.1 State Space and Generator

We have the same state space as in the log economy, but the generator is now modified to:

$$\mathcal{L}_{abe}(c_t, d_t; h_t, P_{at}, P_{bt}, x_t)f(\cdot) = x_t^{-1}\mathcal{L}_e f(\cdot) + \frac{a}{P_{at}} f_a(\cdot) + \frac{1}{\alpha\theta P_{bt}}\left(\alpha\theta b + e^0(1-\alpha) - c_t(\cdot) + h_t\right) f_b(\cdot) \tag{27}$$
$$+ x_t^{-1}\chi\left(f\left(a + \frac{\theta P_{bt}}{(1-\theta)P_{at}} d_t(\cdot), b - d_t(\cdot) - \kappa \mathbb{I}_{d(\cdot)\neq 0}, e\right) - f(\cdot)\right).$$

It now depends on the history of aggregate shocks not only through policy functions and h_t, P_{at}, and P_{bt}, but also through x_t.

An important property of the generator is that it and its adjoint are homogeneous of degree -1 in the last three arguments:

$$\mathcal{L}_{abe}(c, d; h, xP_a, xP_b, x) = x^{-1}\mathcal{L}_{abe}(c, d; h, P_a, P_b, 1), \tag{28}$$

$$\mathcal{L}^\dagger_{abe}(c, d; h, xP_a, xP_b, x) = x^{-1}\mathcal{L}^\dagger_{abe}(c, d; h, P_a, P_b, 1). \tag{29}$$

The expression for h_t must be adjusted to

$$h_t = x_t^{-1}\chi\left(\int \mathbb{I}_{d_t(\cdot)\neq 0} d\mu_t(\cdot)\right)\kappa P_{bt}. \tag{30}$$

3.2 Steady State Equilibrium

The steady state equilibrium can be characterized by the HJB equation:

$$\rho\bar{V}(a, b, e) = \max_{\substack{c\geq 0, \mathbb{I}_{b=0} c \leq (1-\alpha)e^0 + \bar{h}, \\ d\in[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b, b]}} \frac{c^{1-\gamma}}{1-\gamma} + \mathcal{L}_{abe}(c, d; \bar{h}, \bar{P}_a, \bar{P}_b, 1)\bar{V}(a, b, e). \tag{31}$$

The policy functions $\bar{c}$ and $\bar{d}$ are derived from the the HJB, the KFE is:

$$\mathcal{L}^\dagger_{abe}(\bar{c}, \bar{d}; \bar{h}, \bar{P}_a, \bar{P}_b, 1)\bar{\mu} = 0, \tag{32}$$

and $\bar{h} = \chi\int \mathbb{I}_{\bar{d}(\cdot)\neq 0} d\bar{\mu}(\cdot)\kappa\bar{P}_b$. As before, we assume that a steady state equilibrium can be found.

3.3 Competitive Equilibrium with Aggregate Shocks

We conjecture a competitive equilibrium with aggregate shocks that is Markov in (Y, g) and where $h_t^* = \bar{h}$, $P_{at}^* = x_t\bar{P}_a$ and $P_{bt}^* = x_t\bar{P}_b$. Given this conjecture, we can use the properties

of the generator $\mathcal{L}_{aeb}$ in (28) and (29) to write:

$$\mathcal{L}_{abe}\left(c,d;h_t^*,P_{at}^*,P_{bt}^*,x_t\right)=x_t^{-1}\mathcal{L}_{abe}\left(c,d;\bar{h},\bar{P}_a,\bar{P}_b,1\right),$$

and the adjoint:

$$\mathcal{L}_{abe}^{\dagger}\left(c,d;h_t^*,P_{at}^*,P_{bt}^*,x_t\right)=x_t^{-1}\mathcal{L}_{abe}^{\dagger}\left(c,d;\bar{h},\bar{P}_a,\bar{P}_b,1\right).$$

The HJB equation is:

$$\begin{aligned}\rho V\left(a,b,e;Y,g\right)=&\max_{\substack{c\geq 0,\mathbb{I}_{b=0}c\leq(1-\alpha)e^0+\bar{h},\\ d\in\left[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b,b\right]}}Y^{1-\gamma}\frac{c^{1-\gamma}}{1-\gamma}+x\left(Y,g\right)^{-1}\mathcal{L}_{abe}\left(c,d;\bar{h},\bar{P}_a,\bar{P}_b,1\right)V\left(a,b,e;Y,g\right)\\&+\mathcal{L}_{Yg}\left(Y,g\right)V\left(a,b,e;Y,g\right).\end{aligned}\tag{33}$$

We guess and verify that:

$$V\left(a,b,e;Y,g\right)=x\left(Y,g\right)Y^{1-\gamma}\bar{V}\left(a,b,e\right).$$

Replacing our guess into the HJB we get:

$$\begin{aligned}\rho x\left(Y,g\right)Y^{1-\gamma}\bar{V}\left(a,b,e\right)=Y^{1-\gamma}\times&\max_{\substack{c\geq 0,\mathbb{I}_{b=0}c\leq(1-\alpha)e^0+\bar{h}\\ d\in\left[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b,b\right]}}\left(\frac{c^{1-\gamma}}{1-\gamma}+\mathcal{L}_{abe}\left(c,d;\bar{h},\bar{P}_a,\bar{P}_b,1\right)\bar{V}\left(a,b,e\right)\right)\\&+\bar{V}\left(a,b,e\right)\mathcal{L}_{Yg}\left(Y,g\right)\left(x\left(Y,g\right)Y^{1-\gamma}\right).\end{aligned}$$

Using the steady state HJB (31), we see the policy functions are unchanged. Dividing throughout by $\bar{V}\left(a,b,e\right)$, we obtain:

$$\rho x\left(Y,g\right)Y^{1-\gamma}=\rho Y^{1-\gamma}+\mathcal{L}_{Yg}\left(Y,g\right)\left(x\left(Y,g\right)Y^{1-\gamma}\right).\tag{34}$$

We can integrate this expression using the Feynman-Kac formula to obtain:

$$x\left(Y_t,g_t\right)Y_t^{1-\gamma}=\mathbb{E}_t\left[\int_t^{\infty}e^{-\rho(s-t)}\rho Y_s^{1-\gamma}ds|Y_t,g_t\right],$$

which verifies the guess.

The policy functions are the same as in the steady state, $c_t^*=\bar{c}$ and $d_t^*=\bar{d}$, so if we start

at $\mu_0 = \bar{\mu}$ the KFE is:

$$d\mu_t^* = \mathcal{L}_{abe}^{\dagger}\left(c_t^*, d_t^*; h_t^*, P_{at}^*, P_{bt}^*, x_t\right)\mu_t^* dt = x_t^{-1}\mathcal{L}_{abe}^{\dagger}\left(\bar{c}, \bar{d}; \bar{h}, \bar{P}_a, \bar{P}_b, 1\right)\bar{\mu} dt = 0,$$

so $\mu_t^* = \bar{\mu}$. Since the policy functions and the measure are both as in the steady state, we have market clearing and from (30) we confirm that

$$h_t^* = x_t^{-1}\chi\left(\int \mathbb{I}_{\bar{d}(\cdot)\neq 0} d\bar{\mu}(\cdot)\right)\kappa x_t \bar{P}_b = \bar{h}.$$

All that remains is to pin down the returns. For the illiquid asset we have:

$$\mathbb{E}_t\left[\int_t^{\infty} e^{-\int_t^s r_{au}du}(1-\theta)\alpha Y_s ds\right] = x_t \bar{P}_a (1-\theta)\alpha Y_t. \tag{35}$$

Take time derivatives on both sides and divide by $\bar{A}_t$, recalling that $(1-\theta)\alpha Y_t/A_t = 1/(\bar{P}_a x_t) = \bar{r}_a/x_t$, to obtain:

$$\left(r_{at} - \frac{\bar{r}_a}{x_t}\right)dt + \sigma_{at}\cdot dM_t = \left(\frac{\mu_{xt}}{x_t} + g_t\right)dt + \frac{\sigma_{xt}}{x_t}\cdot dM_t,$$

where $\sigma_{xt} = \sigma_x(Y_t, g_t) = x_g(Y,g)\sigma_g(Y,g) + x_Y(Y,g)\sigma_Y(Y,g)$ and μ_{xt} is the drift of x_t. We therefore have $\sigma_{at} = \frac{\sigma_{xt}}{x_t}$. From the definition of x_t we can compute:[15]

$$\frac{\mu_{xt}}{x_t} = \rho - \frac{\rho}{x_t} - (1-\gamma)g_t,$$

so we obtain

$$r_{at} = \rho + \gamma g_t - \frac{\rho - \bar{r}_a}{x_t}.$$

The same procedure works for the liquid asset.

Proposition 3. *Assume $\sigma_Y(Y_t, g_t) = 0$ and that there exists a steady state equilibrium of the CRRA economy $(\bar{c}, \bar{d}, \bar{r}_a, \bar{r}_b, \bar{P}_a, \bar{P}_b, \bar{\mu}, \bar{h})$ with C^1 value function $\bar{V}(a,b,e)$ satisfying the HJB equation (31). Then if $\mu_0 = \bar{\mu}$ and $x(Y_t, g_t)$ as defined in (26) exists and is finite, there exists a competitive equilibrium with:*

$$c_t^*(\cdot) = \bar{c}(\cdot), \quad d_t^*(\cdot) = \bar{d}(\cdot), \quad P_{at}^* = x_t\bar{P}_a, \quad P_{bt}^* = x_t\bar{P}_b, \quad \mu_t^* = \bar{\mu}, \quad h_t^* = \bar{h}.$$

[15] To see this, notice that $\int_0^t e^{-\rho s}Y_s^{1-\gamma}\rho ds + e^{-\rho t}Y_t^{1-\gamma}x_t$ is a martingale. Computing its expected change we obtain the expression for $\frac{\mu_{xt}}{x_t}$.

Expected asset returns are

$$r_{jt} = \rho + \gamma g_t - \frac{\rho - \bar{r}_j}{x_t}, \quad j = a, b, \tag{36}$$

and asset volatility is $\sigma_{at} = \sigma_{bt} = \frac{\sigma_x(Y_t, g_t)}{x(Y_t, g_t)}$.

As in the log case, this result allow us to construct the competitive equilibrium with aggregate shocks assuming the initial distribution $\mu_0 = \bar{\mu}$. In Section 6 we extend this result to construct the competitive equilibrium with aggregate shocks for any initial distribution μ_0.

Discussion of the CRRA Economy

The CRRA economy allows for exact aggregation in spite of the time-varying price-dividend ratios because the state-dependent idiosyncratic labor income process and trading opportunities prevent redistribution in response to aggregate shocks. With $\gamma > 1$ interest rates fall more than one-to-one with growth g_t. If the labor income generator was invariant to the cycle, as in the log economy, the onset of a recession (low g_t) would redistribute from those whose labor income is front-loaded to those whose labor income is backloaded. Unemployed households would gain, in present value terms, with the start of a recession (relative to employed households). Scaling the generator by x_t^{-1} as in (24) prevents this redistribution: This scaling slows down the idiosyncratic shock process when g_t falls, which means that currently-unemployed (or more broadly currently-low-labor-income) households bear a disproportionate share of the reduction in future labor income. This exactly compensates for the lower discounting on future labor income. Thus, the model features cyclicality in labor income risk but no cyclicality in the net-present-value-of-labor-income risk.

But the time-varying price-dividend ratios do change the supply of liquidity and therefore affect the spread between liquid and illiquid assets. Since with $\gamma > 1$ interest rates fall more than one to one with growth, the onset of a recession raises the supply of liquidity relative to the needs derived from labor income and trading risk. As a result, the spread $s_t = (\bar{r}_a - \bar{r}_b)/x_t$ shrinks during recessions, when x_t is large. Exact aggregation with aggregate shocks depends on three elements adjusting in the same way: price-dividend ratios, the generator for idiosyncratic labor income, and trading frictions. This can be seen in the inverse homogeneity property of the $\mathcal{L}_{aeb}$ generator in (28).[16]

[16]Trading frictions become worse during recessions, $x_t^{-1}\chi$, which raises the spread. With constant χ the spread would fall even more.

4 Risk Premia and Zero-Beta Rates

Up to this point we've assumed that aggregate consumption is locally deterministic. In this section we bring consumption volatility back in and study risk premia. This is important because the motivating Fact 2 is that the aggregate consumption Euler equation works when we use the zero-beta rate, the expected return of equities after controlling for risk premia.

The main result in this section is that, even though agents face uninsurable idiosyncratic risk and trading frictions, asset prices nonetheless satisfy a simple Consumption CAPM. We show this in two steps. First, we assume that aggregate consumption may be volatile, $\sigma_Y(Y_t, g_t) > 0$, and show how the competitive equilibrium is modified. Then we introduce zero-net-supply derivatives to complete the market and price any cashflow, and show that the derivatives are not traded and the Consumption CAPM holds.

Step 1: Adding aggregate volatility. With $\sigma_Y(Y_t, g_t)$, relative to the model in Section 3, we need to modify the aggregate generator to account for the volatility of Y and the covariance of the shocks to Y and g:

$$\mathcal{L}_{Yg} f(\cdot) = \begin{bmatrix} gY & \mu_g(Y,g) \end{bmatrix} \cdot \nabla_{gY} f(\cdot) + \frac{1}{2} tr[\sigma(Y,g) \nabla^2_{gY} f(\cdot) \sigma(Y,g)^T],$$

where ∇_{gY} and ∇^2_{gY} are the gradient and Hessian with respect to (g, Y), and $\sigma(Y,g) = \begin{bmatrix} \sigma_g(Y,g) & \sigma_Y(Y,g) \end{bmatrix}$.The derivation in Section 3 is unchanged except for the computation of asset returns off of (35).

Proposition 4. *Assume $\sigma_Y(Y_t, g_t) > 0$ and there exists a steady state equilibrium of the CRRA economy $(\bar{c}, \bar{d}, \bar{r}_a, \bar{r}_b, \bar{P}_a, \bar{P}_b, \bar{\mu}, \bar{h})$ with C^1 value function $\bar{V}(a, b, e)$ satisfying the HJB equation (31). Then if $\mu_0 = \bar{\mu}$ and $x(Y_t, g_t)$ as defined in (26) exists and is finite, there exists a competitive equilibrium with*

$$c^*_t(\cdot) = \bar{c}(\cdot), \quad d^*_t(\cdot) = \bar{d}(\cdot), \quad P^*_{at} = x_t \bar{P}_a, \quad P^*_{bt} = x_t \bar{P}_b, \quad \mu^*_t = \bar{\mu}, \quad h^*_t = \bar{h}.$$

Expected asset returns are

$$r_{jt} = \rho + \gamma g_t - (\gamma - 1) \frac{\gamma}{2} \sigma^2_{Yt} + \gamma \frac{\sigma_{xt}}{x_t} \cdot \sigma_{Yt} - \frac{\rho - \bar{r}_j}{x_t}, \quad j = a, b, \tag{37}$$

and asset volatility is $\sigma_{at} = \sigma_{bt} = \frac{\sigma_x(Y_t, g_t)}{x(Y_t, g_t)} + \sigma_Y(Y_t, g_t)$.

Step 2: Completing asset markets. Second, we introduce zero-net supply derivatives into each account, with unit loading on each element of M_t and premia π_{jt}. This completes

the market within each account, allowing us to price any cashflow. We will show that $\pi_{jt} = \pi_j(Y_t, g_t) = \gamma\sigma_Y(Y_t, g_t)$ and there is no trading in the derivatives, so that the equilibrium is not affected at all by the completion of the market.[17]

Let δ^a_{it} and δ^b_{it} be household i's derivatives position in the illiquid and liquid accounts, as a fraction of the balance in each account. The budget constraints (6) and (7) are now:

$$dA_{it} = (r_{at} + \delta^a_{it} \cdot \pi_{at})\, A_{it}dt + D_{it}dN_{it} + A_{it}\,(\sigma_{at} + \delta^a_{it}) \cdot dM_t,$$

$$dB_{it} = \left(r_{bt}B_{it} + B_{it}\delta^b_{it} \cdot \pi_{bt} + e^0_{it}(1-\alpha)Y_t - C_{it} + H_t\right) dt - (D_{it} + \kappa \mathbb{I}_{D_{it}\neq 0} B_t)\, dN_{it} + B_{it}\left(\sigma_{bt} + \delta^b_{it}\right) \cdot dM_t.$$

The laws of motion of a_{it} and b_{it}, in the absence of a trading opportunity, are now:

$$a_{it} = a_{it}\left(\frac{1}{P_{at}} + \delta^a_{it} \cdot (\pi_{at} - \sigma_{at})\right) dt + a_{it}\delta^a_{it} \cdot dM_t,$$

$$db_{it} = \frac{1}{\alpha\theta P_{bt}}\left(\alpha\theta b_{it} + b_{it}\delta^b_{it} \cdot (\pi_{bt} - \sigma_{at}) + e^0_{it}(1-\alpha) - c_{it} + h_t\right) dt + b_{it}\delta^b_{it} \cdot dM_t.$$

Define $\delta = \begin{bmatrix} a\delta^a \\ b\delta^b \end{bmatrix}$. The generator $\mathcal{L}(c, d, \delta; h, P_a, P_b, x, \pi, \sigma_a, \sigma_b)$ for (a, b, e, Y, g), defined for arbitrary test function f is:

$$\begin{aligned}
\mathcal{L}f = {} & \mathcal{L}_{abe}(c, d; h, P_a, P_b, x)f + \mathcal{L}_{Yg}f \\
& + (a\delta^a) \cdot (\pi_a\,(Y, g) - \sigma_a\,(Y, g))\, f_a + \left(a\delta^b\right) \cdot (\pi_b\,(Y, g) - \sigma_b\,(Y, g))\, f_b \\
& + tr[\delta^T \cdot \nabla_{ab}\nabla_{gY} f \cdot \sigma(Y, g)] + \frac{1}{2} tr[\delta^T \cdot \nabla^2_{ab} f \cdot \delta],
\end{aligned}$$

where ∇_{ab} and ∇^2_{ab} denote the gradient and Hessian with respect to (a, b). The new terms in the second line account for the modified drift of a and b, and the third line accounts for the covariation of the marginal utility of a and b with Y and g, and the second order covariation of the marginal utilities of a and b.

The HJB equation for the value function $V(a, b, e; Y, g)$ is

$$\rho V = \max_{\substack{c\geq 0, \delta, \\ d\in[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b, b]}} Y^{1-\gamma}\frac{c^{1-\gamma}}{1-\gamma} + \mathcal{L}V.$$

With the same guess as before, $V(a, b, e; Y, g) = x(Y, g)Y^{1-\gamma}\bar{V}(a, b, e)$, and applying the generator $\mathcal{L}$ to V, the HJB is the same as with $\delta^a = \delta^b = \vec{0}$, plus the following terms on the

[17] We allow households to continuously trade derivatives even in the absence of a trading opportunity. Since we prove that there is no trading in the derivatives, a hypothetical restriction of derivatives trading to the arrival of trading opportunities would not bind.

right hand side:

$$x\left(Y,g\right)Y^{1-\gamma}\bar{V}_a\left(a,b,e\right)\times\left[\left(a\delta^a\right)\cdot\left(\pi_a\left(Y,g\right)-\sigma_a\left(Y,g\right)\right)+\left(b\delta^b\right)\cdot\left(\pi_b\left(Y,g\right)-\sigma_b\left(Y,g\right)\right)\right]$$
$$+x\left(Y,g\right)Y^{1-\gamma}\bar{V}_a\left(a,b,e\right)\times\left[\left(a\delta^a\right)\cdot\left(\frac{\sigma_x(Y,g)}{x(Y,g)}+\left(1-\gamma\right)\sigma_Y\left(Y,g\right)\right)+\left(b\delta^b\right)\cdot\left(\frac{\sigma_x(Y,g)}{x(Y,g)}+\left(1-\gamma\right)\sigma_Y\left(Y,g\right)\right)\right]$$
$$+\frac{1}{2}x\left(Y,g\right)Y^{1-\gamma}tr[\delta^T\cdot\nabla^2_{ab}\bar{V}\left(a,b,e\right)\cdot\delta].$$

Using $\sigma_a(Y,g)=\frac{\sigma_x(Y,g)}{x(Y,g)}+\sigma_Y(Y,g)$ and $\pi_a(Y,g)=\pi_b(Y,g)=\gamma\sigma_Y(Y,g)$, the first and second lines of this expression cancel out. The third line is negative as long as $\bar{V}(a,b,e)$ is concave in (a,b) for all e, and equal to zero if $\delta^a=\delta^b=\vec{0}$. Thus, there is no trading in the derivatives in either account, and the competitive equilibrium is therefore unchanged by the introduction of these derivatives.

Our economy was carefully constructed to ensure there are no redistributive effects of aggregate shocks. It should not come as a surprise that, starting from this initial point, agents have no desire to trade derivatives on aggregate risk; they have already shared this risk perfectly, despite their inability to insure each other against idiosyncratic risks.

A central object of interest are the zero-beta rates, the expected returns of each asset after controlling for risk premia. In the steady state economy there are no risk premia and therefore $\bar{r}^0_a=\bar{r}_a$ and $\bar{r}^0_b=\bar{r}_b$. With aggregate shocks, we must remove the risk premium from the return of each asset to recover the zero-beta rate, e.g. $r^0_{at}=r_{at}-\pi_{at}\left(\frac{\sigma_{xt}}{x_t}+\sigma_{Yt}\right)$.

Proposition 5 (Consumption CAPM)**.** *Assume $\bar{V}\left(a,b,e\right)$ is concave in (a,b), and that the conditions of Proposition 4 hold. Then the price of risk is $\pi_j\left(Y,g\right)=\gamma\sigma_Y\left(Y,g\right)$ for $j=a,b$. The zero-beta rates satisfy:*

$$r^0_{jt}=\underbrace{\rho+\gamma g_t-\left(\gamma+1\right)\frac{\gamma}{2}\sigma^2_{Yt}}_{\textit{representative-agent Euler eq.}}-\frac{\rho-\bar{r}^0_j}{x_t}. \tag{38}$$

Notice that the first part of the expression for the zero-beta rate is simply the aggregate consumption Euler equation in a representative agent model. In the case without volatility in aggregate consumption, $\sigma_Y(Y_t,g_t)=0$, studied in Section 3, there is no risk premium so asset returns are zero-beta returns, $r^0_{jt}=r_{jt}$, and expression (38) simplifies to (36).

The additional term $\frac{\rho-\bar{r}^0_j}{x_t}$ reflects the benefit the asset offers as insurance against idiosyncratic risk. In the steady state model, this benefit is what leads to a the difference between the rate of time preference and the interest rate, $\rho-\bar{r}^0_j$. In the model with aggregate risk, this difference is scaled by x_t^{-1}, reflecting changes in the scale of liquidity needs relative to liquidity supply.

It is tempting, but incorrect, to think of this term as arising from the interaction of precautionary savings and aggregate risk, along the lines of Constantinides and Duffie (1996). As those authors show, when the magnitude of precautionary savings effects are correlated with aggregate risks, those aggregate risks will carry a price, creating departures from the Consumption CAPM. This is not what is happening in our model; the Consumption CAPM holds exactly, and aggregate risks have (by design) no redistributive effects, and therefore do not drive changes in the relative marginal utilities of consumption across agents. That is, for the purpose of obtaining aggregation, we have shut down the forces that animate the results of Constantinides and Duffie (1996). Instead, in our model, assets have value as insurance against binding borrowing constraints (in the spirit of Aiyagari (1994)), and this value fluctuates with the state of the economy.

5 Quantitative Evaluation

In this section we evaluate our model quantitatively. We first derive a sufficient statistic expression for asset returns that allow us to calibrate the model abstracting from microeconomic details. We then ask whether the model can match our motivating facts. As equations (37) make clear, this will depend on the behavior of the price-dividend ratio x_t, so we examine how x_t needs to behave for our model to fit the facts, and whether this fits the evidence on price-dividend ratios. By working directly with x_t, we sidestep the need to fully spell out the stochastic process for (Y_t, g_t). Lastly, we study the implications of our quantitative exercise for classic puzzles in macro-finance.

5.1 A sufficient statistic expression for asset returns.

To simplify our exercise, we will assume σ_Y is constant and define the "effective" discount rate

$$\tilde{\rho} = \rho + \gamma \mathbb{E}[g_t] - (\gamma + 1)\frac{\gamma}{2}\sigma_Y^2.$$

Under these assumptions, we can rearrange (38) to obtain the following expression:

$$r_{jt}^0 = \mathbb{E}\left[r_{jt}^0\right] + \gamma\left(g_t - \mathbb{E}\left[g_t\right]\right) - \left(\tilde{\rho} - \mathbb{E}\left[r_{jt}^0\right]\right) \times \left(\frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]} - 1\right). \tag{39}$$

The advantage of this expression is that, insofar as we can observe r_{jt}^0, g_t, and x_t, the only free parameters are the preference parameters $\tilde{\rho}$ and γ. All the microeconomic forces in the model (idiosyncratic shocks, trading frictions costs, etc.) matter for asset prices only through

their effect on the average zero-beta liquid and illiquid rates. Any combination of parameters that produces the same averages will generate the same macroeconomic implications. Note also that, up to a linearization,

$$\frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]} - 1 \approx \ln\left(x_t^{-1}\right) - \ln\left(\mathbb{E}\left[x_t^{-1}\right]\right).$$

5.2 Basic calibration.

We use the following calibration targets:

- Liquid assets: we use MZM (zero-maturity monetary assets). This includes cash, checking deposits, saving deposits and money market fund shares. Their weighted average nominal return is just under half the federal funds rate, and moves almost linearly with it.[18] The mean annualized real return is approximately $\mathbb{E}[r_{bt}^0] = -1.5\%$.[19]

- Illiquid assets: We adopt the view as in Di Tella et al. (2023) that stocks are illiquid, consistent with the evidence on marginal propensities to consume out of capital gains. We set the mean real return to $\mathbb{E}[r_{at}^0] = 8.5\%$ annual.[20] The mean spread is $\mathbb{E}[s_t] = \mathbb{E}[r_{at}^0] - \mathbb{E}[r_{bt}^0] = 10\%$.

- Mean growth is $\mathbb{E}[g_t] = 1.5\%$. We set the standard deviation of expected growth to $std(g_t) = 0.5\%$. This means that aggregate consumption growth is somewhat but not very predictable. Assuming a volatility of realized consumption growth of 1.22% (as in Campbell and Cochrane (1999)), it implies an $R^2 \approx 0.17$ in a regression predicting consumption growth using g_t.

- We want the aggregate consumption Euler equation to hold for r_{at}^0 with an intertemporal elasticity of 0.2 (Fact 2). We set $\gamma = 5$ and $\rho - (\gamma+1)\frac{\gamma}{2}\sigma_Y^2 = 1\%$. This yields $\tilde{\rho} = \mathbb{E}\left[r_{at}^0\right] = 8.5\%$, and the aggregate consumption Euler equation for r_{at}^0 fits perfectly.

- We want the aggregate consumption Euler equation to fail for r_{bt}^0 (Fact 3). Consider

[18] This implies that Euler equations using the MZM rate or the federal funds rate will fit equally poorly.

[19] This results from the following calculation. We assume that the nominal return on cash plus checking deposits (M1) is zero; the nominal return on savings deposits (M2 minus M1 minus small denomination time deposits) is 0.43 times the federal funs rate, following Kurlat (2019); and the nominal return on money market mutual funds is equal to the federal funds rate. This yields a weighted average nominal return on MZM between 1973 and 2020 of 2.39%, which corresponds to a -1.47% real return, which we round to -1.5%.

[20] This is close to the point estimate in the main specification of Di Tella et al. (2023) (8.3%). We round to 8.5% to make the exercise that follows easier to follow.

the linear projection of the dividend-price ratio onto g_t,

$$\frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]} = 1 + \beta\left(g_t - \mathbb{E}_t\left[g_t\right]\right) + \epsilon_t, \quad \mathbb{E}\left[\epsilon_t\right] = \mathbb{E}\left[\epsilon_t g_t\right] = 0.$$

If we plug this into the expression for r_{bt}, (39), and use that $\tilde{\rho} = \mathbb{E}[r_{at}^0]$, we obtain:

$$r_{bt}^0 - \mathbb{E}\left[r_{bt}^0\right] = \left(\gamma - \beta\mathbb{E}\left[s_t\right]\right) \times \left(g_t - \mathbb{E}\left[g_t\right]\right) - \mathbb{E}\left[s_t\right]\epsilon_t. \tag{40}$$

If the dividend-price ratio x_t^{-1} was perfectly correlated with g_t (that is, $\epsilon_t = 0$), then we would recover an aggregate consumption Euler equation for r_{bt}, but with a different intertemporal elasticity, $\left(\gamma - \beta\mathbb{E}\left[s_t\right]\right)^{-1}$. For the aggregate consumption Euler equation to fail for r_{bt} we need significant volatility in ϵ_t. In other words, we need the consumption-wealth ratio to have a component that is uncorrelated to current expected consumption growth. This is consistent with a central fact in asset pricing: dividend-price ratios are volatile and only weakly predict cashflow growth (Campbell and Shiller, 1988; Cochrane, 2008).

For the behavior of x_t we calibrate $std(x_t^{-1}/\mathbb{E}[x_t^{-1}]) = 26\%$ and $\beta = 20$. The standard deviation of the dividend-price ratio of 26% is in line with the calibration of the consumption-wealth ratio in Campbell and Cochrane (1999) and a little below the log volatility of the dividend-price ratio for stocks, while $\beta = 20$ implies a coefficient in the inverse regression (g_t on $\frac{x_t^{-1}}{\mathbb{E}[x_t^{-1}]}$) of 0.0074, in line with Cochrane (2008).[21]

With this calibration, the Euler equation for r_{bt}^0 fails. A regression of r_{bt}^0 on expected consumption growth g_t, would have an R^2 of 28%. That is, most of the variance of r_{bt}^0 is not explained by g_t. In contrast, the Euler equation for r_{at}^0 fits perfectly by construction, with an R^2 of 100%. In terms of realized consumption growth instead of expected consumption growth, the R^2 of a regression predicting consumption growth with r_{bt} is 0.044, while using r_{at} it is 0.17, broadly consistent with the evidence of Di Tella et al. (2023).[22]

5.3 Asset Pricing Puzzles

Let us now consider the implications of our calibration for three classic asset pricing puzzles: the equity premium puzzle, the risk-free rate puzzle, and the equity volatility puzzle, and

[21] The coefficient in the inverse regression is $cov\left(g_t, \frac{x_t^{-1}}{\mathbb{E}[x_t^{-1}]}\right) / var\left(\frac{x_t^{-1}}{\mathbb{E}[x_t^{-1}]}\right) = \beta var\left(g_t\right) / \left(\beta^2 var(g_t) + var(\epsilon_t)\right)$, and $var(\epsilon_t) = var\left(\frac{x_t^{-1}}{\mathbb{E}[x_t^{-1}]}\right) - \beta^2 var(g_t)$.

[22] Separate regressions of one-year ahead real consumption growth on the real T-bill rate and real zero-beta rate in the main specification/sample of Di Tella et al. (2023) yield R^2 values of 3% and 16%, respectively.

for return predictability. These asset pricing puzzles have been extensively discussed in the literature; our definitions will follow Weitzman (2007). To fix ideas, let us imagine that $\ln(\left(x_t^{-1}/\mathbb{E}\left[x_t^{-1}\right]\right))$ follows an Ornstein-Uhlenbeck (OU) process,

$$d\ln\left(\frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]}\right) = \kappa\left(\mu - \ln\left(\frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]}\right)\right)dt + \sigma dM_{1,t},$$

with $\kappa = -\ln(0.91)$ and $\frac{\sigma^2}{2\kappa} = (0.26)^2$. This process is the continuous time analog of an AR(1) process with an annual auto-correlation of 0.91, and it has a steady state standard deviation of 26% (as in our calibration above). The annual auto-correlation of 0.91 matches the observed auto-correlation of the price-dividend ratio.

Equity premium puzzle. Our model, calibrated to match the zero-beta rate $\mathbb{E}[r_{at}^0]$ and MZM safe rate $\mathbb{E}[r_{bt}^0]$, will roughly match the average returns of equity over the MZM safe rate provided that the risk premium $\mathbb{E}[r_{at} - r_{at}^0]$ is small, in which case it will also be consistent with the finding of a flat security market line. That is, our calibration requires a small, not large, risk premium to match the observed equity premium, because the liquidity premium is substantial. The risk premium in our model is the one implied by the consumption CAPM, and hence will be small under standard assumptions. Put another way, there is an equity premium in our model, but there is no puzzle, provided that one calibrates to $\mathbb{E}[r_{at}^0]$ and $\mathbb{E}[r_{bt}^0]$.

Equity Volatility Puzzle. Stock returns are much more volatile than, and not particularly correlated with, consumption growth. In our model, the instantaneous volatility of the illiquid asset is $\frac{\sigma_x(Y,g)}{x(Y,g)} + \sigma_Y(Y,g)$, and the instantaneous volatility of consumption growth is $\sigma_Y(Y,g)$. In our calibration, the instantaneous volatility of the illiquid asset (a consumption claim) is $\sigma = 11.3\%$, which is an order of magnitude larger than the volatility of annual consumption growth. Most of the volatility in returns comes from the volatility of the dividend-price ratio. Moreover, if we assume (as is realistic) that there is only a small correlation between consumption growth and expected future consumption growth, our illiquid asset returns will have a low correlation with realized consumption growth. These findings would only be accentuated in a model that distinguishes between consumption and dividends. We conclude that our model will qualitatively match the facts of the equity volatility puzzle provided that it is calibrated to generate a volatile and persistent dividend-price ratio.

Risk-free rate puzzle. Our model is calibrated to match $\mathbb{E}[r_{bt}^0]$. Matching the level of the risk-free rate (the aspect of the puzzle emphasized by Weitzman (2007)) is straightforward,

provided that the model generates a large liquidity premium, which is the case in standard calibrations of two-account heterogeneous agents models (Kaplan and Violante (2022)). Our calibration implies a volatility of r_{bt} of 2.8%, which should be compared to the volatility of the MZM rate of 1.94%, in the data (using realized inflation over the past 12 months as a measure of inflation expectations).

Return Predictability. Valuation ratios (in our context, x_t^{-1}) predict the future excess return of the market (r_{at}) over treasury bills (r_{bt}^0), as documented in e.g. Campbell and Shiller (1988) and Cochrane (2011). Most models that attempt to explain return predictability do so by linking valuation ratios with risk premia. In our framework, a different mechanism exists. Even if risk premia are zero ($r_{at} = r_{at}^0$), we will have, by (39),

$$r_{at}^0 - r_{bt}^0 = \left(\mathbb{E}\left[r_{at}^0\right] - \mathbb{E}\left[r_{bt}^0\right]\right) \frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]}.$$

Consider a regression of four years' worth of expected excess returns on the dividend-price ratio (x_t^{-1}), take as given the OU process assume above. It follows that (up to a linearization):

$$\int_t^{t+4} \left(r_{as}^0 - r_{bs}^0\right) ds = 3.3 \times \left(\mathbb{E}\left[r_{at}^0\right] - \mathbb{E}\left[r_{bt}^0\right]\right) \times \frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]} + \nu_{t+h}, \quad \mathbb{E}\left[v_{t+h}\right] = \mathbb{E}\left[\nu_{t+h} x_t^{-1}\right] = 0,$$

where the coefficient of 3.33 accounts for the persistence of x_t^{-1}.[23] With our calibration of $\mathbb{E}[r_{at}^0] - \mathbb{E}[r_{bt}^0] = 10\%$, the implied regression coefficient is 0.33. Up to the linearization that equates the log dividend-price ratio with $\frac{x_t^{-1}}{\mathbb{E}[x_t^{-1}]} - 1$, this is exactly the coefficient observed in the data (as reported in Wachter (2013)). This should not be a surprise– our model is calibrated to match the persistence of the log dividend-price ratio and the weak ability of that ratio to predict dividend growth, and hence by the Campbell-Shiller approximation will necessarily lead to return predictability (Cochrane (2008)).

Our calibration can explain return predictability regressions entirely through the ability of valuation ratios to predict the spread between liquid and illiquid assets, even in the complete absence of risk premia (consistent with the VAR evidence Di Tella et al. (2023)). Moreover, this property will hold in any calibration that matches empirical evidence on (i) the spread between illiquid zero-beta rates and liquid rates, (ii) the persistence of valuation ratios, and (iii) a low correlation between the slope of the security market line and valuation ratios. The last of these ensures that risk premia (relative to zero-beta rates) are largely unrelated to

[23] If the OU mean reversion parameter is $-\ln(0.91)$, then $\mathbb{E}_t\left[\int_t^{t+4} \ln\left(\frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]}\right) ds\right] = \frac{(0.91^4-1)}{\ln(0.91)} \ln\left(\frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]}\right) \approx 3.3\left(1 + \frac{x_t^{-1}}{\mathbb{E}\left[x_t^{-1}\right]}\right)$.

valuation ratios, and hence do not substantially affect the above regression.

5.4 Remaining Quantitative Questions

We have argued that, properly calibrated, our model can match both our motivating facts about Euler equations and a set of classic asset pricing facts, given our assumptions on the preference parameters (ρ, γ). Because our model generates tractable, closed-form asset pricing expressions, we have been able to show this without specifying any of the details of the steady-state heterogeneous agent model. At the same time, we have imposed assumptions on the variance, covariance, and persistence of (x_t^{-1}, g_t), without explicitly constructing a process for (Y_t, g_t) that would give rise to these dynamics.

This leaves open three questions for future research. First, can a two-account heterogeneous agent model with this (ρ, γ) match evidence on MPCs, given realistic assumptions on trading frictions and individual income processes? The evidence of Kaplan and Violante (2022) suggests that this is possible. In particular, those authors usually estimate a ρ that is close to $\bar{r}_a^0$, which is exactly what our model requires for an aggregate Euler equation to hold for the illiquid asset.[24] Note, however, that our calibration requires $\gamma = 5$, not log utility, and that the steady-state equilibrium interest rates differ substantially from their empirical means, as a consequence of trend growth and aggregate risk:

$$\bar{r}_j^0 = \rho + \frac{\tilde{\rho} - \mathbb{E}[r_{jt}^0]}{\mathbb{E}[x_t^{-1}]}.$$

Second, what kind of process for (Y_t, g_t) is required to generate a consumption-wealth ratio with the desired volatility, persistence, and predictability properties? The behavior of the consumption-wealth ratio in the model coincides with the expression for the consumption-wealth ratio in a representative-agent model, an object that has been extensively studied in the asset-pricing literature. Our calibration requires a strong relationship between short-run expected consumption growth (g_t) and the dividend-price ratio (x_t), which will occur in models with persistent growth. At the same time, it requires substantial variation in x_t that is unrelated to g_t; persistent shocks to the volatility of g_t are a natural source of such variation. The long-run risks model of Bansal and Yaron (2004) has both of these features, and hence (at least in terms of functional forms) is potentially consistent with what our model requires. But Bansal and Yaron (2004) (and existing asset-pricing models more generally) generate volatility in x_t^{-1} and low predictability of consumption growth by generating a large and volatile risk premium. Our model points in a different direction: the risk premium is

[24] This is not a coincidence, and arises from the presence of relatively wealthy agents in the calibration.

small, in line with Consumption CAPM and a flat securities market line. Instead, what is needed is a model where the zero-beta rate (the mean of the SDF) is volatile.

Third, we should note again that our model intentionally shuts down redistributive effects for the purpose of obtaining aggregation. We expect that these effects have important implications for asset prices but exactly what those are remains an open question. Our exercise should be understood as developing a tractable benchmark and then evaluating that benchmark, rather than as a complete quantitative theory of macroeconomics and asset pricing.

6 Transition Dynamics

In Sections 2 and 3 we constructed competitive equilibria with aggregate shocks for the case where the cross-sectional distribution starts at its steady state value, $\mu_0 = \bar{\mu}$. Here we extend those results for an arbitrary initial distribution, μ_0,. We construct competitive equilibria that feature both aggregate shocks and transition dynamics.

This exercise is useful in two distinct ways. First, as we will show, we can "add on" aggregate risk to arbitrary transition dynamics. There is an extensive literature studying transition dynamics in the presence of incomplete markets, and our results show one way in which the insights of this literature can be extended to consider the effects of aggregate risk. Second, for our particular purposes, incorporating transition dynamics can allow us to consider one-time unanticipated redistributive shocks in our framework while at the same time allowing for aggregate risk. We leave a further exploration of these possibilities to future research.

6.1 Deterministic path

Consider a aggregate-deterministic economy starting with an arbitrary (Y_0, g_0) and μ_0 but no aggregate shocks $M_t = 0$, and associated paths $\bar{Y}(t)$, $\bar{g}(t)$, and $\bar{x}(t)$. Assume that there exists a competitive equilibrium,

$$\bar{c}(\cdot, t), \quad \bar{d}(\cdot, t), \quad \bar{P}_a(t), \quad \bar{P}_b(t), \quad \bar{r}_a(t), \quad \bar{r}_b(t), \quad \bar{\mu}(t), \quad \bar{h}(t).$$

and $\bar{\sigma}_a(t) = \bar{\sigma}_b(t) = 0$. The generator $\mathcal{L}_{abe}$ is as in (27). The value function is $\bar{V}(a, b, e, t)$, with HJB equation

$$\rho \bar{V}(a,b,e,t) = \max_{\substack{c\geq 0, \mathbb{I}_{b=0} c \leq (1-\alpha)e^0 + \bar{h}(t), \\ d \in \left[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b, b\right]}} \bar{Y}(t)^{1-\gamma}\frac{c^{1-\gamma}}{1-\gamma} + \mathcal{L}_{abe}\left(c, d; \bar{h}(t), \bar{P}_a(t), \bar{P}_b(t), \bar{x}(t)\right)\bar{V}(a,b,e,t) + \bar{V}_t(a,b,e,t \tag{41}$$

The policy functions $\bar{c}(\cdot, t)$ and $\bar{d}(\cdot, t)$ are derived from the HJB equation, the KFE is

$$d\bar{\mu}(t) = \mathcal{L}^{\dagger}_{abe}\left(\bar{c}(\cdot,t), \bar{d}(\cdot,t); \bar{h}(t), \bar{P}_a(t), \bar{P}_b(t), \bar{x}(t)\right)\bar{\mu}(t)dt,$$

with $\bar{\mu}(0) = \mu_0$, and $\bar{h}(t) = \bar{x}(t)^{-1}\chi\left(\int \mathbb{I}_{\bar{d}(\cdot,t)\neq 0} d\bar{\mu}(\cdot,t)\right)\kappa\bar{P}_b(t)$.

The deterministic path is a generalization of the steady state equilibrium we used in Sections 2 and 3. If we start with the initial distribution μ_0 corresponding to the steady state, then the deterministic path coincides with the steady state equilibrium.

6.2 Stochastic Time Change

We will construct the competitive equilibrium using the deterministic path with a stochastic time change. Define the stochastic process τ_t as the solution to the SDE

$$\tau_t = \int_0^t \bar{x}(\tau_s) x_s^{-1} ds,$$

where $x_t = x(Y, g)$ is as in (26). τ is a strictly increasing stochastic process, but the speed is not uniform. Time goes always forward, but at varying speed. It encodes the history of aggregate shocks in terms of deviations from the deterministic path. If shocks actually realize all to zero ($M_t = 0$), then $x_t = \bar{x}(t)$ and $\tau_t = t$. Aggregate shocks that raise x_t relative to the deterministic path (with $\gamma > 1$, a period of low growth relative to the path, such as a recession), are encoded as a slowdown of time τ.

6.3 Competitive Equilibrium with Shocks

We will look for a competitive equilibrium that is Markov in (Y, g, τ) with $h^*_t = \bar{h}(\tau_t)$, $P^*_{at} = \frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t)$ and $P^*_{bt} = \frac{x_t}{\bar{x}(\tau_t)}\bar{P}_b(\tau_t)$. Notice that the deterministic-path objects $\bar{x}(\cdot)$ and $\bar{P}_j(\cdot)$ are evaluated at the stochastic time τ_t instead of the usual t. Using (28), the generator is

$$\mathcal{L}_{abe}(c, d; h^*_t, P^*_{at}, P^*_{bt}, x_t) = \bar{x}(\tau_t)x_t^{-1}\mathcal{L}_{abe}\left(c, d; \bar{h}(\tau_t), \bar{P}_a(\tau_t), \bar{P}_b(\tau_t), \bar{x}(\tau_t)\right). \tag{42}$$

We will also use the infinitesimal generator for Y, g, and τ,

$$\begin{aligned}\mathcal{L}_{Yg\tau}f(\cdot) &= f_Y(\cdot)gY + f_g(\cdot)\mu_g(Y,g) + f_\tau(\cdot)\bar{x}(\tau)x(Y,g)^{-1} \\ &+ \frac{1}{2}f_{gg}(\cdot)\sigma_g(Y,g)^2 + \frac{1}{2}f_{YY}(\cdot)\sigma_Y(Y,g)^2 + f_{gY}(\cdot)\sigma_g(Y,g)^T \cdot \sigma_Y(Y,g).\end{aligned}$$

The value function is $V(a,b,e;Y,g,\tau)$ and the HJB equation (suppressing arguments to avoid clutter)

$$\rho V = \max_{\substack{c\geq 0, \mathbb{I}_{b=0}c\leq(1-\alpha)e^0+\bar{h}(\tau) \\ d\in[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b,b]}} Y^{1-\gamma}\frac{c^{1-\gamma}}{1-\gamma} + \bar{x}(\tau)x\left(Y,g\right)^{-1}\mathcal{L}_{abe}\left(c,d;\bar{h}(\tau),\bar{P}_a(\tau),\bar{P}_b(\tau),\bar{x}(\tau)\right)V + \mathcal{L}_{Yg\tau}V.$$

We guess and verify that

$$V(a,b,e,Y,g,\tau) = \frac{x(Y,g)}{\bar{x}(\tau)}\left(\frac{Y}{\bar{Y}(\tau)}\right)^{1-\gamma}\bar{V}\left(a,b,e,\tau\right).$$

Plugging this into the HJB equation:

$$\begin{aligned}\rho\frac{x(Y,g)}{\bar{x}(\tau)}\left(\frac{Y}{\bar{Y}(\tau)}\right)^{1-\gamma}\bar{V} &= \left(\frac{Y}{\bar{Y}(\tau)}\right)^{1-\gamma} \\ &\times\left\{\max_{\substack{c\geq 0, \mathbb{I}_{b=0}c\leq(1-\alpha)e^0+\bar{h}(\tau) \\ d\in\left[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b,b\right]}} \bar{Y}(\tau)^{1-\gamma}\frac{c^{1-\gamma}}{1-\gamma} + \mathcal{L}_{abe}\left(c,d;\bar{h}(\tau),\bar{P}_a(\tau),\bar{P}_b(\tau),\bar{x}(\tau)\right)\bar{V} + \bar{V}_\tau\right\} \\ &+\bar{V}\mathcal{L}_{Yg\tau}\left[\frac{x(Y,g)}{\bar{x}(\tau)}\left(\frac{Y}{\bar{Y}(\tau)}\right)^{1-\gamma}\right].\end{aligned}$$

Use the deterministic-path HJB to notice the policy functions will be unchanged but evaluated at τ, $c_t^* = \bar{c}(\tau_t)$ and $d_t^* = \bar{d}(\tau_t)$. Plug in (41) and divide by $\bar{V}$ to get:

$$\rho\frac{x(Y,g)}{\bar{x}(\tau)}\left(\frac{Y}{\bar{Y}(\tau)}\right)^{1-\gamma} = \rho\left(\frac{Y}{\bar{Y}(\tau)}\right)^{1-\gamma} + \mathcal{L}_{Yg\tau}\left[\frac{x(Y,g)}{\bar{x}(\tau)}\left(\frac{Y}{\bar{Y}(\tau)}\right)^{1-\gamma}\right], \tag{43}$$

which, expanding the last, term is:

$$\rho\frac{x(Y,g)Y^{1-\gamma}}{\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}} = \rho\frac{Y^{1-\gamma}}{\bar{Y}(\tau)^{1-\gamma}} + \frac{\mathcal{L}_{Yg\tau}x(Y,g)Y^{1-\gamma}}{\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}} - \frac{x(Y,g)Y^{1-\gamma}}{\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}}\frac{\mathcal{L}_{Yg\tau}\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}}{\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}}. \tag{44}$$

To verify our guess we need to make sure this expression is true. First, write $x(Y,g)$ and $\bar{x}(t)$ in the following form

$$x(Y,g)Y^{1-\gamma} = \mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)}\rho Y_s^{1-\gamma}ds|Y_t = Y, g_t = g\right],$$

$$\bar{x}(t)\bar{Y}(t)^{1-\gamma} = \mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)}\rho \bar{Y}(s)^{1-\gamma}ds\right].$$

We can write the HJB equations for each,

$$\mathcal{L}_{Yg\tau}x(Y,g)Y^{1-\gamma} = \rho x(Y,g)Y^{1-\gamma} - \rho Y^{1-\gamma}, \tag{45}$$

$$\nabla_t \bar{x}(t)\bar{Y}(t)^{1-\gamma} = \rho\bar{x}(t)\bar{Y}(t)^{1-\gamma} - \rho\bar{Y}(t)^{1-\gamma}, \tag{46}$$

$$\implies \mathcal{L}_{Yg\tau}\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma} = (\rho\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma} - \rho\bar{Y}(\tau)^{1-\gamma}) \times \bar{x}(\tau)x(Y,g)^{-1}, \tag{47}$$

where the last expression uses the time change. Plugging (45) and (47) into (44) and cancelling terms, we verify our guess.

Since the policy functions are unchanged, the measure is $\mu_t = \bar{\mu}(\tau_t)$. The KFE is

$$d\mu_t = \mathcal{L}_{abe}^\dagger\left(c_t^*, d_t^*; h_t^*, P_{at}^*, P_{bt}^*, x_t\right)\mu_t dt = \underbrace{\mathcal{L}_{abe}\left(\bar{c}(\tau_t), \bar{d}(\tau_t); \bar{h}(\tau_t), \bar{P}_a(\tau_t), \bar{P}_b(\tau_t), \bar{x}(\tau_t)\right)\bar{\mu}(\tau_t)}_{d\bar{\mu}(\tau)/d\tau}\underbrace{\bar{x}(\tau_t)x_t^{-1}dt}_{d\tau_t}. \tag{48}$$

Since both policy functions and μ correspond to the deterministic path evaluated at τ, we also have market clearing, and from (14) we confirm that

$$h_t^* = x_t^{-1}\chi\left(\int \mathbb{I}_{\bar{d}(\cdot,\tau_t)\neq 0}d\bar{\mu}(\cdot,\tau_t)\right)\kappa\bar{x}(\tau_t)^{-1}x_t\bar{P}_b = \bar{h}(\tau_t).$$

All that remains is to pin down the returns. For the illiquid asset

$$\mathbb{E}_t\left[\int_t^\infty e^{-\int_t^s r_{au}du}(1-\theta)\alpha Y_s ds\right] = \frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t)(1-\theta)\alpha Y_t.$$

Take time derivatives and match terms to obtain

$$r_{at}^* = \rho + \gamma g_t - (\gamma-1)\frac{\gamma}{2}\sigma_{Yt}^2 + \gamma\frac{\sigma_{xt}}{x_t}\cdot\sigma_{Yt} - \frac{\bar{x}(\tau_t)}{x_t}(\rho + \gamma\bar{g}(\tau_t) - \bar{r}_a(\tau_t)),$$

$$\sigma_{at} = \frac{\sigma_{xt}}{x_t} + \sigma_{Yt}.$$

The same argument can be applied to liquid assets.

Proposition 6. *Take initial conditions Y_0, g_0, and μ_0 as given, with associated deterministic path $\bar{Y}(t)$, $\bar{g}(t)$, and $\bar{x}(t)$. Assume there exists a deterministic equilibrium of the CRRA economy $(\bar{c}(\cdot,t), \bar{d}(\cdot,t), \bar{r}_a(t), \bar{r}_b(t), \bar{P}_a(t), \bar{P}_b(t), \bar{\mu}(t), \bar{h}(t))$ with C^1 value function $\bar{V}(a,b,e,t)$ satisfying the HJB equation (41). Assume furthermore that $x(Y,g)$ as defined in (26) exists and is finite and that*

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\rho T}\frac{x_T}{\bar{x}(\tau_T)}\left(\frac{Y_T}{\bar{Y}(\tau_T)}\right)^{1-\gamma}\right] = 0.$$

Then there exists a competitive equilibrium with:

$$c_t^*(\cdot) = \bar{c}(\cdot,\tau_t), \quad d_t^*(\cdot) = \bar{d}(\cdot,\tau_t), \quad P_{at}^* = \frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t), \quad P_{bt}^* = \frac{x_t}{\bar{x}(\tau_t)}\bar{P}_b(\tau_t), \quad \mu_t^* = \bar{\mu}(\tau_t), \quad h_t^* = \bar{h}(\tau_t),$$

where asset returns are:

$$r_{jt}^* = \rho + \gamma g_t - (\gamma-1)\frac{\gamma}{2}\sigma_{Yt}^2 + \gamma\frac{\sigma_{xt}}{x_t}\cdot\sigma_{Yt} - \frac{\bar{x}(\tau_t)}{x_t}(\rho + \gamma\bar{g}(\tau_t) - \bar{r}_j(\tau_t)), \quad j\in\{a,b\} \tag{49}$$

and asset volatility is $\sigma_{at} = \sigma_{bt} = \frac{\sigma_x(Y_t,g_t)}{x(Y_t,g_t)} + \sigma_Y(Y_t,g_t)$.

Propositions 3 and 4 can be viewed as special cases of this result, in which $\bar{g}(t) = 0$, $\bar{x}(t) = 1$, and $\bar{\mu}(t)$ is constant an equal to the ergodic population measure.

We can introduce derivatives into each account to dynamically complete the market. An analogous argument as in Section 4 shows there is no trading in the derivatives and a simple Consumption CAPM holds in each account.

Proposition 7 (Consumption CAPM)**.** *Assume $\bar{V}(a,b,e,t)$ is concave in a and b. Then the price of risk is $\pi_j(Y,g) = \gamma\sigma_Y(Y,g)$ for $j = a,b$. The zero-beta rates satisfy for $j = a,b$:*

$$r_{jt}^0 = \underbrace{\rho + \gamma g_t - (\gamma+1)\frac{\gamma}{2}\sigma_{Yt}^2}_{\textit{representative-agent Euler eq.}} - \frac{\bar{x}(\tau_t)}{x_t}(\rho + \gamma\bar{g}(\tau_t) - \bar{r}_j(\tau_t)). \tag{50}$$

6.4 Summary

We have shown that normalized consumption, income, and share holdings $(c_{it}, e_{it}, a_{it}, b_{it})$ will follow the same transition path in the economy with aggregate risk that they would have followed in the absence of aggregate risk. However, the speed at which they follow this transition path it itself stochastic. In particular, when shocks cause aggregate growth to slow down, the speed at which the economy converges towards its steady state also slows down. This is true regardless of the nature of the transition, which could itself be caused by a one-time, unanticipated shock with redistributive consequences.

7 Conclusion

In this paper, we propose a liquidity-based theory of consumption and asset prices. We analytically characterize asset prices in a two-account heterogeneous agent model with uninsured idiosyncratic risk, borrowing constraints, and aggregate risk. The main result is that the trading frictions implied by the high marginal propensity to consume at the household level can explain (1) a zero-beta rate for equities that satisfies an aggregate consumption Euler equation, (2) a safe rate that does not, and (3) volatile price-dividend ratios weakly related to expected consumption growth, and (4) a flat securities-market line. Our methods can be extended to the case of transition dynamics and thus provide analytical results in models that are close to the quantitative models of the existing literature.

References

Sushant Acharya and Keshav Dogra. Understanding hank: Insights from a prank. *Econometrica*, 88(3):1113–1158, 2020.

Sushant Acharya, Edouard Challe, and Keshav Dogra. Optimal monetary policy according to hank. *American Economic Review*, 113(7):1741–1782, 2023.

S Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684, 1994.

Adrien Auclert, Matthew Rognlie, and Ludwig Straub. The intertemporal keynesian cross. 2023.

Ravi Bansal and Wilbur John Coleman. A monetary explanation of the equity premium, term premium, and risk-free rate puzzles. *Journal of political Economy*, 104(6):1135–1171, 1996.

Ravi Bansal and Amir Yaron. Risks for the long run: A potential resolution of asset pricing puzzles. *The journal of Finance*, 59(4):1481–1509, 2004.

Anmol Bhandari, Thomas Bourany, David Evans, and Mikhail Golosov. A perturbational approach for approximating heterogeneous-agent models. Technical report, Working Paper, 2023.

Adrien Bilal. Solving heterogeneous agent models with the master equation. Technical report, National Bureau of Economic Research, 2023.

Florin O Bilbiie. Limited asset markets participation, monetary policy and (inverted) aggregate demand logic. *Journal of economic theory*, 140(1):162–196, 2008.

Fischer Black. Capital market equilibrium with restricted borrowing. *The Journal of business*, 45(3):444–455, 1972.

Fischer Black, Michael C Jensen, Myron Scholes, et al. The capital asset pricing model: Some empirical tests. 1972.

William A Brock and Leonard J Mirman. Optimal economic growth and uncertainty: the discounted case. *Journal of Economic Theory*, 4(3):479–513, 1972.

John Y Campbell and John H Cochrane. By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of political Economy*, 107(2):205–251, 1999.

John Y Campbell and Robert J Shiller. The dividend-price ratio and expectations of future dividends and discount factors. *The Review of Financial Studies*, 1(3):195–228, 1988.

Edouard Challe. Uninsured unemployment risk and optimal monetary policy in a zero-liquidity economy. *American Economic Journal: Macroeconomics*, 12(2):241–283, 2020.

Gabriel Chodorow-Reich, Plamen T. Nenov, and Alp Simsek. Stock market wealth and the real economy: A local labor market approach. *American Economic Review*, 111(5): 1613–57, May 2021.

John H Cochrane. The dog that did not bark: A defense of return predictability. *The Review of Financial Studies*, 21(4):1533–1575, 2008.

John H Cochrane. Presidential address: Discount rates. *The Journal of finance*, 66(4): 1047–1108, 2011.

George M Constantinides and Darrell Duffie. Asset pricing with heterogeneous consumers. *Journal of Political economy*, 104(2):219–240, 1996.

Sebastian Di Tella, Benjamin M Hébert, Pablo Kurlat, and Qitong Wang. The zero-beta interest rate. Technical report, National Bureau of Economic Research, 2023.

William Diamond. Safety transformation and the structure of the financial system. *The Journal of Finance*, 75(6):2973–3012, 2020.

Andrea Frazzini and Lasse Heje Pedersen. Betting against beta. *Journal of financial economics*, 111(1):1–25, 2014.

Lars Peter Hansen and Kenneth J Singleton. Generalized instrumental variables estimation of nonlinear rational expectations models. *Econometrica: Journal of the Econometric Society*, pages 1269–1286, 1982.

Harrison Hong and David A Sraer. Speculative betas. *The Journal of Finance*, 71(5): 2095–2144, 2016.

Tullio Jappelli and Luigi Pistaferri. The consumption response to income changes. *Annual Review of Economics*, 2(1):479–506, 2010.

David S Johnson, Jonathan A Parker, and Nicholas S Souleles. Household expenditure and the income tax rebates of 2001. *American Economic Review*, 96(5):1589–1610, 2006.

Greg Kaplan and Giovanni L Violante. A model of the consumption response to fiscal stimulus payments. *Econometrica*, 82(4):1199–1239, 2014.

Greg Kaplan and Giovanni L Violante. The marginal propensity to consume in heterogeneous agent models. *Annual Review of Economics*, 14:747–775, 2022.

Greg Kaplan, Giovanni L. Violante, and Justin Weidner. The wealthy hand-to-mouth. *Brookings Papers on Economic Activity*, pages 77–138, 2014.

Greg Kaplan, Benjamin Moll, and Giovanni L Violante. Monetary policy according to hank. *American Economic Review*, 108(3):697–743, 2018.

Dirk Krueger and Hanno Lustig. When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)? *Journal of Economic Theory*, 145(1):1–41, 2010.

Pablo Kurlat. Deposit spreads and the welfare cost of inflation. *Journal of Monetary Economics*, 2019.

Ricardo Lagos. Asset prices and liquidity in an exchange economy. *Journal of Monetary Economics*, 57(8):913–930, 2010.

Morten O Ravn and Vincent Sterk. Macroeconomic fluctuations with hank & sam: An analytical approach. *Journal of the European Economic Association*, 19(2):1162–1202, 2021.

Jessica A Wachter. Can time-varying risk of rare disasters explain aggregate stock market volatility? *The Journal of Finance*, 68(3):987–1035, 2013.

Martin L Weitzman. Subjective expectations and asset-return puzzles. *American Economic Review*, 97(4):1102–1130, 2007.

Iván Werning. Incomplete markets and aggregate demand. 2015.

A Appendix

A.1 Proofs

Lemma 1

Proof. The realized return on the illiquid asset is the same in the aggregate and for the individual household, so:

$$\begin{aligned} r_{at}dt + \sigma_{at}dM_t &= \frac{dA_t}{A_t} + \frac{\alpha(1-\theta)Y_t}{A_t} \\ &= \frac{dA_{it}}{A_{it}} \end{aligned} \tag{51}$$

From the definition of a_{it}, it follows that:

$$da_{it} = \left(\frac{dA_{it}}{A_{it}} - \frac{dA_t}{A_t}\right) a_{it}$$

Replacing 51 and using the definitions of P_{at}:

$$\begin{aligned} da_{it} &= \frac{\alpha(1-\theta)Y_t}{A_t} a_{it} \\ &= \frac{a_{it}}{P_{at}} \end{aligned}$$

Similarly, for the liquid asset:

$$\begin{aligned} r_{bt}dt + \sigma_{bt}dM_t &= \frac{dB_t}{B_t} + \frac{\alpha\theta Y_t}{B_t} \\ &= \frac{dB_{it}}{B_{it}} + c_{it}\frac{Y_t}{B_{it}} - e^0_{it}\left(1-\alpha\right)\frac{Y_t}{B_{it}} - h_t\frac{Y_t}{B_{it}}. \end{aligned}$$

and therefore:

$$\begin{aligned} db_{it} &= \left(\frac{dB_{it}}{B_{it}} - \frac{dB_t}{B_t}\right) b_{it} \\ &= \left(-c_{it}\frac{Y_t}{B_{it}} + e^0_{it}\left(1-\alpha\right)\frac{Y_t}{B_{it}} + h_t\frac{Y_t}{B_{it}} + \frac{\alpha\theta Y_t}{B_t}\right) b_{it} \\ &= \frac{-c_{it} + e^0_{it}\left(1-\alpha\right) + h_t + \alpha\theta b_{it}}{\theta\alpha P_{bt}} \end{aligned}$$

□

Proposition 2

Proof. This is a special case of Proposition 4 with $\gamma \to 1$ and $\sigma_Y(Y_t, g_t) = 0$. □

Proposition 3

Proof. This is a special case of Proposition 4 with $\sigma_Y(Y_t, g_t) = 0$. □

Proposition 4

Proof. It remains to show the derivation of asset returns and the verification of the HJB equation.

(1) We will compute returns for the illiquid asset A. An analogous argument works for the liquid asset B. Start with

$$\mathbb{E}_t\left[\int_t^\infty e^{-\int_t^s r_{au}du}(1-\theta)\alpha Y_s ds\right] = x_t \bar{P}_a (1-\theta)\alpha Y_t.$$

Taking time-derivatives we obtain

$$\begin{aligned}\left(r_{at}x_t\bar{P}_a(1-\theta)\alpha Y_t - (1-\theta)\alpha Y_t\right)dt + \bar{A}_t\sigma_{at}\cdot dM_t = x_t\bar{P}_a(1-\theta)\alpha Y_t\left(\frac{\mu_{xt}}{x_t} + g_t + \sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t}\right)dt \\ + \bar{A}_t\left(\frac{\sigma_{xt}}{x_t} + \sigma_{Yt}\right)\cdot dM_t.\end{aligned}$$

Matching terms we obtain

$$\sigma_{at} = \frac{\sigma_{xt}}{x_t} + \sigma_{Yt},$$

$$r_{at}x_t\bar{P}_a(1-\theta)\alpha Y_t - (1-\theta)\alpha Y_t = x_t\bar{P}_a(1-\theta)\alpha Y_t\left(\frac{\mu_{xt}}{x_t} + g_t + \sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t}\right).$$

Divide throughout by $x_t\bar{P}_a(1-\theta)\alpha Y_t$ and use $\bar{P}_a^{-1} = \bar{r}_a$ to get

$$r_{at} = \frac{\bar{r}_a}{x_t} + \frac{\mu_{xt}}{x_t} + g_t + \sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t}.$$

To compute μ_{xt}/x_t, start with the observation that

$$\int_0^t e^{-\rho s}Y_s^{1-\gamma}\rho ds + e^{-\rho t}Y_t^{1-\gamma}x_t = \mathbb{E}_t\left[\int_0^\infty e^{-\rho s}Y_s^{1-\gamma}\rho ds\right]$$

is a martingale. It's expected change is therefore

$$e^{-\rho t}Y_t^{1-\gamma}\rho + e^{-\rho t}Y_t^{1-\gamma}x_t\left(-\rho + (1-\gamma)g_t - \gamma(1-\gamma)\frac{1}{2}\sigma_{Yt}^2 + \frac{\mu_{xt}}{x_t} + (1-\gamma)\sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t}\right) = 0,$$

and solving for $\frac{\mu_{xt}}{x_t}$ we obtain

$$\frac{\mu_{xt}}{x_t} = \rho - \frac{\rho}{x_t} - (1-\gamma)g_t + \gamma(1-\gamma)\frac{1}{2}\sigma_{Yt}^2 - (1-\gamma)\sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t}. \tag{52}$$

Plug this into the expression for r_{at} above,

$$r_{at} = \rho + \gamma g_t - (\gamma-1)\frac{\gamma}{2}\sigma_{Yt}^2 + \gamma\sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t} - \frac{\rho - \bar{r}_a}{x_t},$$

as desired. An analogous computation works for the illiquid asset.

(2) For the verification of the HJB equation we need to show that for any feasible policy for C_{it} and D_{it}, the terminal term

$$\lim_{T\to\infty}\mathbb{E}_t\left[e^{-\rho T}x(Y_T,g_T)Y_T^{1-\gamma}\bar{V}(a_{iT},b_{iT},e_{iT})\right] \geq 0,$$

with equality for the optimal policy.

Start with the definition of x_t,

$$x(Y_t,g_t)Y_t^{1-\gamma} = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}\rho Y_s^{1-\gamma}ds|Y_t,g_t\right]$$

which we assume exists and is finite for all (Y_t,g_t). Since $Y_t > 0$ we have that $x(Y_t,g_t) > 0$. Decompose it into

$$x(Y_t,g_t)Y_t^{1-\gamma} = \mathbb{E}_t\left[\int_t^T e^{-\rho(s-t)}\rho Y_s^{1-\gamma}ds|Y_t,g_t\right] + \mathbb{E}\left[e^{-\rho(T-t)}x(Y_T,g_T)Y_T^{1-\gamma}|Y_t,g_t\right],$$

or rearranging,

$$\mathbb{E}\left[e^{-\rho(T-t)}x(Y_T,g_T)Y_T^{1-\gamma}|Y_t,g_t\right] = x(Y_t,g_t)Y_t^{1-\gamma} - \mathbb{E}_t\left[\int_t^T e^{-\rho(s-t)}\rho Y_s^{1-\gamma}ds|Y_t,g_t\right].$$

Now take limits on both sides as $T\to\infty$

$$\lim_{T\to\infty}\mathbb{E}\left[e^{-\rho(T-t)}x(Y_T,g_T)Y_T^{1-\gamma}|Y_t,g_t\right] = x(Y_t,g_t)Y_t^{1-\gamma} - \lim_{T\to\infty}\mathbb{E}_t\left[\int_t^T e^{-\rho(s-t)}\rho Y_s^{1-\gamma}ds|Y_t,g_t\right]$$

Use the monotone convergence theorem on the last term to obtain

$$\lim_{T\to\infty}\mathbb{E}\left[e^{-\rho(T-t)}x(Y_T,g_T)Y_T^{1-\gamma}|Y_t,g_t\right]=x(Y_t,g_t)Y_t^{1-\gamma}-\underbrace{\mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}\rho Y_s^{1-\gamma}ds|Y_t,g_t\right]}_{=x(Y_t,g_t)Y_t^{1-\gamma}}=0.$$

Since $\gamma>1$, we have $\bar{V}(a,b,e)<0$. Because e_{it} and Y_t are bounded below away from zero and the borrowing constraints do not allow negative assets, $A_{i,t},B_{it}\geq 0$, each household's consumption is also bounded away from zero and so the steady-state value function is bounded below, $\bar{V}(a_{it},b_{it},e_{it})\geq-\kappa$. We then can construct the bounds

$$0\geq\lim_{T\to\infty}\mathbb{E}_t\left[e^{-\rho T}x(Y_T,g_T)Y_T^{1-\gamma}\bar{V}(a_{iT},b_{iT},e_{iT})\right]\geq-\lim_{T\to\infty}\mathbb{E}_t\left[e^{-\rho T}x(Y_T,g_T)Y_T^{1-\gamma}\kappa\right]=0,$$

for any feasible policy, including the optimal one. □

Proposition 5

Proof. See the main text for the bulk of the proof. The expression for the zero-beta rates comes from applying the definition of the zero-beta rate to the expression in (37) and using $\pi_{jt}=\gamma\sigma_{Yt}$. □

Proposition 6

Proof. Relative to the proof of Proposition 4 we need to (1) recompute asset returns and (2) verify the HJB equation.

(1) For asset returns, we will work with the illiquid asset. An analogous argument works for the liquid one. Start with

$$\mathbb{E}_t\left[\int_t^\infty e^{-\int_t^s r_{au}du}(1-\theta)\alpha Y_s ds\right]=\frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t)(1-\theta)\alpha Y_t.$$

Taking time derivatives we get

$$\left(r_{at}\frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t)(1-\theta)\alpha Y_t-(1-\theta)\alpha Y_t\right)dt+\bar{A}_t\sigma_{at}\cdot dM_t$$

$$=\frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t)(1-\theta)\alpha Y_t\left(\frac{\mu_{xt}}{x_t}+g_t+\sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t}+\frac{\nabla_\tau\left[\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}\right]}{\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}}\times\bar{x}(\tau_t)x_t^{-1}\right)dt+\bar{A}_t\left(\frac{\sigma_{xt}}{x_t}+\sigma_{Yt}\right)\cdot dM_t$$

Matching terms we get

$$\sigma_{at}=\frac{\sigma_{xt}}{x_t}+\sigma_{Yt}$$

and

$$r_{at}\frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t)(1-\theta)\alpha Y_t - (1-\theta)\alpha Y_t$$

$$= \frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t)(1-\theta)\alpha Y_t\left(\frac{\mu_{xt}}{x_t} + g_t + \sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t} + \frac{\nabla_\tau\left[\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}\right]}{\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}} \times \bar{x}(\tau_t)x_t^{-1}\right).$$

Divide throughout by $\frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t)(1-\theta)\alpha Y_t$

$$r_{at} - (\frac{x_t}{\bar{x}(\tau_t)}\bar{P}_a(\tau_t))^{-1} = \left(\frac{\mu_{xt}}{x_t} + g_t + \sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t} + \frac{\nabla_\tau\left[\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}\right]}{\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}} \times \bar{x}(\tau_t)x_t^{-1}\right).$$

The expression for μ_{xt}/x_t is the same as in (52). Plug in and rearrange to get

$$r_{at} = \rho + \gamma g_t - (\gamma-1)\frac{\gamma}{2}\sigma_{Yt}^2 + \gamma\sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t} - \frac{\bar{x}(\tau_t)}{x_t}\left(\frac{\rho}{\bar{x}(\tau_t)} - \frac{1}{\bar{P}_a(\tau_t)} - \frac{\nabla_\tau\left[\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}\right]}{\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}}\right).$$

Now we compute $\nabla_\tau\left[\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}\right]/\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}$. From the definition of $\bar{P}_a(t)$ we get

$$\bar{r}_a(t)\bar{P}_a(t)(1-\theta)\alpha\bar{Y}(t) - (1-\theta)\alpha\bar{Y}(t) = \bar{P}_a(t)(1-\theta)\alpha\bar{Y}(t)\left(\frac{\bar{P}_a'(t)}{\bar{P}_a(t)} + \bar{g}(t)\right),$$

and rearranging

$$\frac{\bar{P}_a'(t)}{\bar{P}_a(t)} = \bar{r}_a(t) - \frac{1}{\bar{P}_a(t)} - \bar{g}(t).$$

Likewise, from the definition of $\bar{x}(t)$ we get

$$\rho\bar{x}(t)\bar{Y}(t)^{1-\gamma} - \rho\bar{Y}(t)^{1-\gamma} = \bar{x}(t)\bar{Y}(t)^{1-\gamma}\left(\frac{\bar{x}'(t)}{\bar{x}(t)} + (1-\gamma)\bar{g}(t)\right),$$

$$\frac{\bar{x}'(t)}{\bar{x}(t)} = \rho - \frac{\rho}{\bar{x}(t)} - (1-\gamma)\bar{g}(t).$$

Putting the two things together and evaluating at $t = \tau_t$, we have

$$\frac{\nabla_\tau\left[\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}\right]}{\frac{\bar{P}_a(\tau_t)}{\bar{x}(\tau_t)}} = \bar{r}_a(\tau_t) - \frac{1}{\bar{P}_a(\tau_t)} - \bar{g}(\tau_t) - \rho + \frac{\rho}{\bar{x}(\tau_t)} + (1-\gamma)\bar{g}(\tau_t),$$

$$= \bar{r}_a(\tau_t) - \frac{1}{\bar{P}_a(\tau_t)} - \rho + \frac{\rho}{\bar{x}(\tau_t)} - \gamma\bar{g}(\tau_t).$$

Now plug this into the expression for r_{at} above,

$$r_{at} = \rho+\gamma g_t-(\gamma-1)\frac{\gamma}{2}\sigma_{Yt}^2+\gamma\sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t}-\frac{\bar{x}(\tau_t)}{x_t}\left(\frac{\rho}{\bar{x}(\tau_t)} - \frac{1}{\bar{P}_a(\tau_t)} - \bar{r}_a(\tau_t) + \frac{1}{\bar{P}_a(\tau_t)} + \rho - \frac{\rho}{\bar{x}(\tau_t)} + \gamma\bar{g}(\tau_t)\right),$$

and simplify to obtain the desired expression

$$r_{at} = \rho + \gamma g_t - (\gamma-1)\frac{\gamma}{2}\sigma_{Yt}^2 + \gamma\sigma_{Yt}\cdot\frac{\sigma_{xt}}{x_t} - \frac{\bar{x}(\tau_t)}{x_t}\left(\rho + \gamma\bar{g}(t) - \bar{r}_a(t)\right).$$

(2) To verify the HJB equation we follow a similar strategy to that in the proof of Proposition 4. We still have that $\bar{V}(a_{it}, b_{it}, e_{it})$ is negative and bounded for any feasible strategy, $\bar{V}(a_{it}, b_{it}, e_{it}) \geq -\kappa$. Using the assumption that

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\rho T}\frac{x_T}{\bar{x}(\tau_T)}\left(\frac{Y_T}{\bar{Y}(\tau_T)}\right)^{1-\gamma}\right] = 0,$$

we can construct the bound

$$0 \geq \lim_{T\to\infty} \mathbb{E}\left[e^{-\rho T}\frac{x_T}{\bar{x}(\tau_T)}\left(\frac{Y_T}{\bar{Y}(\tau_T)}\right)^{1-\gamma}\bar{V}(a_{it}, b_{it}, e_{it})\right] \geq -\lim_{T\to\infty} \mathbb{E}\left[e^{-\rho T}\frac{x_T}{\bar{x}(\tau_T)}\left(\frac{Y_T}{\bar{Y}(\tau_T)}\right)^{1-\gamma}\kappa\right] = 0.$$

□

Proposition 7

Proof. Analogous to proof of 5. □

A.2 Alternative formulations of fixed cost of rebalancing

The baseline model has the rebalancing fees expressed in terms of the liquid asset B_t and rebated to household lump-sum. Here we explore some alternative formulations of this friction.

Rebalancing fees as monopoly rents for intermediaries. A valid interpretation of the baseline model is that the fees represent monopoly power on the part of unmodeled financial intermediaries. These profits are then distributed to households who own the intermediary. Under this interpretation measured GPD would not correspond to Y_t, but rather to $GDP_t =$

$Y_t + H_t$. Fortunately, since $H_t = \bar{h} Y_t$, we see that GDP is just a constant multiple of Y_t, $GDP_t = Y_t(1 + \bar{h})$.

Tradeable claims on the intermediary. The interpretation of the baseline model above assumes that ownership of the intermediary cannot be traded. That's why the profits, the aggregate rebalancing fees H_t, appear as a lump-sum transfer. We can take this idea a step further and treat the intermediary the same way as the rest of GDP, and its profits as tradeable. Now the budget constraints must be modified to remove the lump-sum transfer H_t,

$$dB_{it} = \left(r_{bt} B_{it} + e^0_{it} (1 - \alpha) Y_t - C_{it}\right) dt - \left(D_{it} + \mathbb{I}_{D_{it} \neq 0} \kappa B_t\right) dN_{it} + B_{it} \sigma_{bt} \cdot dM_t, \tag{53}$$

but instead they show up as part of the value of assets.

$$A_t = \mathbb{E}_t \left[\int_t^\infty e^{-\int_t^s r_{au} du} (1 - \theta)(\alpha Y_s + H_s) ds \right], \tag{54}$$

$$B_t = \mathbb{E}_t \left[\int_t^\infty e^{-\int_t^s r_{bu} du} \theta (\alpha Y_s + H_s) ds \right]. \tag{55}$$

We assume that H_t can be used to back liquid and illiquid assets in the same proportion as the rest of capital income αY_t. The price-dividend ratios are now

$$P_{at} = \frac{A_t}{(1 - \theta)(\alpha Y_t + H_t)}, \quad P_{bt} = \frac{B_t}{\theta(\alpha Y_t + H_t)}.$$

The model is different because households are really able to trade the profits coming from the intermediary. The idiosyncratic generator is now

$$\begin{aligned} \mathcal{L}_{abe}(c_t, d_t; h_t, P_{at}, P_{bt}, x_t) f(\cdot) = x_t^{-1} \mathcal{L}_e f(\cdot) + \frac{a}{P_{at}} f_a(\cdot) + \frac{1}{P_{bt}} \left(b + \frac{e^0(1 - \alpha) - c_t(\cdot)}{\theta(\alpha + h_t)} \right) f_b(\cdot) \\ + x_t^{-1} \chi \left(f\left(a + \frac{\theta P_{bt}}{(1 - \theta) P_{at}} d_t(\cdot), b - d_t(\cdot) - \kappa \mathbb{I}_{d(\cdot) \neq 0}, e \right) - f(\cdot) \right), \end{aligned} \tag{56}$$

and the constraint on consumption when $b = 0$ is $c \leq (1 - \alpha) e^0$. The steady state equilibrium will therefore be different from the baseline model. However, we can still construct the equilibrium with aggregate shocks with $P_{at} = x_t \bar{P}_a$, $P_{bt} = x_t \bar{P}_b$, and $h_t = \bar{h}$. Under this guess $\mathcal{L}_{abe}$ is still homogeneous of degree -1 in the last three arguments, and therefore the same argument in Section 3 applies. Proposition 3 applies to this variant of the model.

Rebalancing fees as real resource cost. We have thus far specified the rebalancing fees in terms of unit of the liquid asset B_t. The rebalancing fees therefore do not show up in the resource constraint. A third alternative is to specify them as a real cost in consumption units (but still scaling with the value of the liquid asset B_t). The fees are therefore neither rebated nor capitalized in the value of assets, but instead show up in the resource constraint. That is, the budget constraints for the liquid asset is (53) but the value of assets is given by the original (8) and (9). The resource constraint is modified to

$$\int_0^1 C_{it} = Y_t - H_t.$$

The idiosyncratic generator is now

$$\mathcal{L}_{abe}(c_t, d_t; P_{at}, P_{bt}, x_t)f(\cdot) = x_t^{-1}\mathcal{L}_e f(\cdot) + \frac{a}{P_{at}} f_a(\cdot) + \frac{1}{P_{bt}}\left(b + \frac{e^0(1-\alpha) - c_t(\cdot)}{\theta\alpha}\right) f_b(\cdot) \quad (57)$$

$$+x_t^{-1}\chi\bigg(f\Big(a + \frac{\theta P_{bt}}{(1-\theta)P_{at}} d_t(\cdot), b - d_t(\cdot) - \kappa\mathbb{I}_{d(\cdot)\neq 0}, e\Big) - f(\cdot)\bigg).$$

and the resource constraint in normalized variables

$$\int_{\text{supp}(\mu_t^*)} c_t^*(a,b,e)d\mu_t^*(a,b,e) = 1 - h_t.$$

The steady state equilibrium is therefore different, but we can still construct the equilibrium with aggregate shocks with $P_{at} = x_t\bar{P}_a$, $P_{bt} = x_t\bar{P}_b$, and $h_t = \bar{h}$. Under this guess $\mathcal{L}_{abe}$ is still homogeneous of degree -1 in the last three arguments, and therefore the same argument in Section 3 applies. Proposition 3 applies to this variant of the model as well.

Rebalancing fees as a utility cost. Finally, we could also think of rebalancing fees in terms of utility cost. To obtain exact aggregation, we scale the utility cost with $Y_t^{-\gamma}B_t$. The utility cost scales with the value of the liquid asset in aggregate consumption units. The fee disappears from the budget constraint,

$$dB_{it} = \left(r_{bt}B_{it} + e_{it}^0\left(1-\alpha\right)Y_t - C_{it}\right)dt - D_{it}dN_{it} + B_{it}\sigma_{bt}\cdot dM_t, \quad (58)$$

and the resource constraint

$$\int_0^1 C_{it}di = Y_t.$$

The generator is therefore modified to

$$\mathcal{L}_{abe}(c_t, d_t; P_{at}, P_{bt}, x_t)f(\cdot) = x_t^{-1}\mathcal{L}_e f(\cdot) + \frac{a}{P_{at}}f_a(\cdot) + \frac{1}{P_{bt}}\left(b + \frac{e^0(1-\alpha) - c_t(\cdot)}{\theta\alpha}\right)f_b(\cdot) \quad (59)$$

$$+x_t^{-1}\chi\left(f\left(a + \frac{\theta P_{bt}}{(1-\theta)P_{at}}d_t(\cdot), b - d_t(\cdot), e\right) - f(\cdot)\right).$$

Instead, an extra term appears on the HJB equation. In the steady state

$$\rho\bar{V}(a,b,e) = \max_{\substack{c\geq 0, \mathbb{I}_{b=0}c\leq(1-\alpha)e^0 \\ d\in[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b, b]}} \frac{c^{1-\gamma}}{1-\gamma} - \chi\kappa\alpha\theta\bar{P}_b\mathbb{I}_{d\neq 0} + \mathcal{L}_{abe}(c,d;\bar{P}_a,\bar{P}_b,1)\bar{V}(a,b,e). \quad (60)$$

In the equilibrium with aggregate shocks with $P_{at} = x_t\bar{P}_a$, $P_{bt} = x_t\bar{P}_b$

$$\rho V(a,b,e;Y,g) = \max_{\substack{c\geq 0, \mathbb{I}_{b=0}c\leq(1-\alpha)e^0 \\ d\in\left[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b, b\right]}} Y^{1-\gamma}\frac{c^{1-\gamma}}{1-\gamma} - x(Y,g)^{-1}\chi\kappa Y^{-\gamma}x(Y,g)\alpha\theta Y\bar{P}_b\mathbb{I}_{d\neq 0} \quad (61)$$

$$+ x(Y,g)^{-1}\mathcal{L}_{abe}\left(c,d;\bar{P}_a,\bar{P}_b,1\right)V(a,b,e;Y,g) \quad (62)$$

$$+ \mathcal{L}_{Yg}(Y,g)V(a,b,e;Y,g).$$

With the same guess as in the baseline case,

$$V(a,b,e;Y,g) = x(Y,g)Y^{1-\gamma}\bar{V}(a,b,e),$$

we get:

$$\rho x(Y,g)Y^{1-\gamma}\bar{V}(a,b,e) = \max_{\substack{c\geq 0, \mathbb{I}_{b=0}c\leq(1-\alpha)e^0 \\ d\in\left[-a\frac{1-\theta}{\theta}\bar{P}_a/\bar{P}_b, b\right]}} Y^{1-\gamma}\left(\frac{c^{1-\gamma}}{1-\gamma} - \chi\kappa\alpha\theta\bar{P}_b\mathbb{I}_{d\neq 0} + \mathcal{L}_{abe}\left(c,d;\bar{P}_a,\bar{P}_b,1\right)\bar{V}(a,b,e)\right)$$

$$+ \bar{V}(a,b,e)\mathcal{L}_{Yg}(Y,g)\left(x(Y,g)Y^{1-\gamma}\right).$$

and the rest of the proof proceeds as in the baseline case in Section 3. Proposition 3 applies to this variant of the model as well, with $\bar{h} = 0$.